Scattering Amplitudes and Loop Calculations in Quantum Field Theory

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by

Ayush Singh

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National Institute of Science Education and Research (NISER)
at Bhimpur, Padanpur, P.O. Jatani,
Khurda – 752050, Odisha, India

Declaration

Signature of Student:
Date:

The work reported in the thesis entitled *"Scattering Amplitudes and Loop Calculations in Quantum Field Theory"* was carried out under my supervision, in the School of Physical Sciences at NISER, Bhubaneswar, India.

Signature of Thesis Supervisor:
School:
Date:

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Abstract

THIS REPORT GIVES AN OVERVIEW of my work in quantum field theory and gauge theory over the last year. First part of this report summarizes the machinery of renormalized perturbation theory for quantum field theories. Calculations of renormalizing Z-factors, beta functions, anomalous dimensions of the field, and anomalous dimension of mass are done for different quantum field theories — scalar field theories, Yukawa theory, electrodynamics, and nonabelian gauge theory — to one loop order in perturbation theory.

The second part of this report goes over modern techniques for computing tree level scattering amplitudes in gauge theory. For computation of scattering amplitudes in nonabelian gauge theory, in particular, spinor helicity formalism, twistor variables, and colour ordering are introduced, and traditional Feynman rules are translated into this new language. With these new Feynman rules, colour ordered amplitudes are computed for QCD processes like $qq \to qq$, $q\bar{q} \to q\bar{q}$ and most importantly, $gg \to gg$.

On-shell recursion relations for tree level processes are also introduced, which let us completely bypass the business of drawing and computing Feynman diagrams, and give us tools to completely determine higher point amplitudes in terms of lower point amplitudes. Finally, to demonstrate the power of these methods an efficient proof of the famous Parke–Taylor formula for maximally helicity violating (MHV) amplitudes is constructed using BCFW recursion.

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Chapter 1

Introduction

First part of this project was dedicated to learning the quantization and renormalization of quantum field theories, and getting used to the kinds of calculations involved in perturbative QFT. During this phase of the project I did calculations of beta functions and anomalous dimensions for different quantum field theories to one loop order.

In the second part, I learned the spinor helicity formalism and modern techniques for computing scattering amplitudes in nonabelian gauge theory. In particular, I learned how to compute colour ordered tree level amplitudes of certain perturbative QCD processes, using both traditional Feynman rules translated to the spinor helicity language and on-shell BCFW recursion relation. Finally, I have done a proof of the famous Parke–Taylor formula using BCFW recursion.

Loops

After a field theory has been quantized and Feynman rules been derived, next to leading order terms in perturbation theory (the so called *radiative corrections*) turn out to be ill-defined: they involve integration over an undetermined loop momentum, and this integral typically diverges in the ultraviolet limit. It is sometimes possible to isolate and absorb these divergences by introducing an *ultraviolet cutoff* and renormalizing fields and couplings so that observable quantities like scattering amplitudes and cross sections remain finite despite the Lagrangian containing terms that depend on the formally divergent UV cutoff. If it is possible to absorb divergences to all orders in perturbation theory by renormalizing fields and parameters (or by adding a finite number of new terms to the Lagrangian), the quantum field theory is said to be *renormalizable*.

Renormalized fields and parameters in a renormalizable theory include a dependence on a parameter called the renormalization scale, and requiring bare (unrenormalized) fields and parameters to be independent of this scale gives important qualitative information about the theory. For example, requiring the bare coupling to be independent of the renormalization scale leads to an expression for the *beta function* which tells us how the strength of interactions varies with the energy scale, or in the jargon of quantum field theory, how the coupling *runs*.

In this report we start with some of the simplest quantum field theories— ϕ^3 theory in six spacetime dimensions and ϕ^4 theory in four spacetime dimensions—to illustrate how divergent integrals are dealt with using dimensional regularization and how fields and parameters are renormalized, without additional complications that multiple interactions, fermions or gauge fields involve.

Yukawa theory is the first theory we encounter in which additional interaction terms have to be added to make the theory renormalizable by absorbing divergences from certain vertex functions. Complications due to multiple different interactions in calculation of beta functions and anomalous dimensions are dealt with in detail in this section, and the results are carried over to gauge theories. Other than the multiple couplings, and learning to handle Dirac spinors and gamma matrices, there are no additional calculational difficulties.

After this, we apply the techniques learned in simpler theories to compute the famous *vacuum polarization*, *electron self-energy* and *vertex correction* diagrams in quantum electrodynamics. Using the divergent part of renormalizing factors, we calculate the QED beta function, corrections to the scaling dimension of fermion and gauge fields, and anomalous dimension of fermion mass. We also do these calculations in *scalar electrodynamics*—a theory of complex scalars coupled to the electromagnetic field. Apart from two more interaction vertices leading to a larger number of diagrams, scalar electrodynamics poses no additional calculational difficulties.

Finally, we compute loop diagrams, beta functions and anomalous dimensions in nonabelian gauge theory coupled to spinors and scalars. There are many technical issues involving gauge fixing and quantisation of gauge fields, but we do not dwell on these in this report, instead preferring to start from a gauge fixed Lagrangian and coupling gauge fields to matter by the gauge principle. When the gauge group is SU(3), results of this section reproduce the famous negative beta function of quantum chromodynamics, which indicates that QCD is asymptotically free.

Amplitudes

With traditional Feynman rules for nonabelian gauge theory, vertex terms get extremely complicated extremely quickly, to wit: even the tree level amplitude for $gg \to gg$ process is very difficult to compute in perturbation theory.

For massless particles, the *spinor helicity formalism* and *twistor variables* help avoid the labour involved in simplifying expressions for amplitudes, and in conjunction with Gervais–Neveu gauge and colour ordering, they lead to simple and compact final expressions. But computing amplitudes from Feynman rules written in the language of twistor variables, despite being much simpler than traditional methods, gets cumbersome for higher point amplitudes. For example, for 5-, 6-, or 7-gluon amplitudes, one has to compute 10, 38, and 154 Feynman diagrams respectively.

The approach of *on-shell recursion relations* uses the power of complex analysis to factor higher-point on-shell amplitudes into a product of complex-shifted, lower-point on-shell amplitudes. These methods avoid the whole process of drawing and computing Feynman diagrams. BCFW (Britto, Cachazo, Feng, Witten) recursion relations, in particular, provide a very efficient inductive proof of the Parke–Taylor formula, and also a way to construct more general higher point amplitudes using the MHV amplitude.

In Part 2, spinor helicity methods and twistor variables are introduced, and to demonstrate these techniques tree level amplitude for Compton scattering is computed. Next, we deal with various four point amplitudes in nonabelian gauge theory. Colour-ordered amplitudes for QCD processes $qq \to qq$, $q\bar{q} \to gg$ and $gg \to gg$ (and their crossing related cousins) are computed and expressed in terms of twistor variables. BCFW recursion is introduced in Chapter 7 and is used to compute the four point gluon amplitude using various three point amplitudes. Finally, the BCFW recursion is used to prove the Parke–Taylor formula by mathematical induction over the number of external gluons.

In appendices, we list important formulae used throughout this report, and compile Feynman rules for all theories studied in the main text for reference.

Part I
Loops

CHAPTER 2

Scalar Field Theories

WE START WITH TWO OF THE SIMPLEST quantum field theories to see the machinery of renormalized perturbation theory in action, unencumbered by additional complications that multiple interactions, fermions and gauge fields entail. In particular, we shall see *Feynman's trick* of converting a product of propagators to an integral, and the method of *dimensional regularization* as a way to introduce an ultraviolet cutoff.

Yukawa theory is the first quantum field theory we encounter in which additional interaction terms have to be added to make it renormalizable. Multiple interactions lead to many one loop diagrams that have to be evaluated, and also to certain complications in the computation of beta functions and anomalous dimensions. Results derived in this section are also used for electromagnetism and nonabelian gauge theory coupled to scalars.

Calculations for ϕ^3 theory in this chapter have been adapted from SREDNICKI.

2.1 Cubic self-interaction

Scalar field theory with a ϕ^3 self-interaction is described by a Lagrangian of the form

$$L = -\frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_M M^2 \phi^2 + \frac{1}{3!} Z_\varkappa \varkappa \phi^3 + Y\phi. \tag{2.1}$$

By dimensional analysis, we notice that in d dimensions, the cubic self-coupling has mass dimension of

$$[\varkappa] = \frac{1}{2}(6-d). \tag{2.2}$$

Interactions in quantum field theory are the most interesting when the coupling is dimensionless, therefore we shall study ϕ^3 theory in six spacetime dimensions.

The Lagrangian can be organized into *free*, *interacting*, and *counterterm* pieces as follows,

$$L_0 = -\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} M^2 \phi^2, \tag{2.3}$$

$$L_1 = \frac{1}{3!} Z_\varkappa \varkappa \phi^3 + L_{\text{ct}}, \tag{2.4}$$

$$L_{\text{ct}} = -\frac{1}{2}(Z_\phi - 1)\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}(Z_M - 1) M^2 \phi^2 + Y\phi. \tag{2.5}$$

In what follows, we shall study one-by-one the kinds of one loop corrections that occur with a ϕ^3 interaction.

2.1.1 Cancelling tadpoles

For the validity of the LSZ formula, vacuum expectation value of the field is required to be zero, $\langle 0|\phi(x)|0\rangle = 0$. Graphically, the vacuum expectation value equals the sum of diagrams with a single source, with one source removed (Figure 2.1).

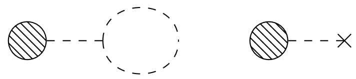


Figure 2.1: Tadpoles in scalar ϕ^3 theory.

To the lowest order in $\varkappa$, we have

$$\langle 0|\phi(x)|0\rangle = \left(iY + \frac{1}{2}(i\varkappa)\frac{1}{i}\Delta(0)\right)\int d^6y\frac{1}{i}\Delta(x-y) + O(\varkappa^3). \tag{2.6}$$

Therefore, to have $\langle 0|\phi(x)|0\rangle = 0$, we must choose

$$Y = \frac{1}{2}i\varkappa\Delta(0). \tag{2.7}$$

However, the integral

$$\Delta(0) = \int \frac{d^6k}{(2\pi)^6}\frac{1}{k^2+M^2-i\epsilon}, \tag{2.8}$$

is unbounded. The divergence can be isolated by analytically continuing the integral to $d = 6-\epsilon$ dimensions,

$$\Delta(0) = \tilde{\mu}^{\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+M^2-i\epsilon}, \tag{2.9}$$

where the parameter $\tilde{\mu}$ with dimensions of mass has been introduced to keep the dimension of $\Delta(0)$ unchanged.

In the integral $\int d^dk$, we can view the integral over k^0 as an integral over a contour that goes from $-\infty$ to $+\infty$. We can do a *Wick rotation* $k^0 \to ik^0$, so that the contour runs from $-i\infty$ to $+i\infty$. As long as the contour does not pass over any poles while making the rotation, the value of the integral remains unchanged. We can now replace k by a Euclidean vector $\bar{k}$, given by $\bar{k}_j = k_j$ for $j = 1,\ldots,d-1$, and $\bar{k}_d = ik^0$. We note that

$$k^2 = \bar{k}^2 = \bar{k}_1^2 + \cdots + \bar{k}_d^2, \tag{2.10}$$

and $d^dk = id^d\bar{k}$. As a result, we have

$$\tilde{\mu}^{\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+M^2-i\epsilon} = i\tilde{\mu}^{\epsilon}\int \frac{d^d\bar{k}}{(2\pi)^d}\frac{1}{\bar{k}^2+M^2-i\epsilon}. \tag{2.11}$$

where the integral now is over a Euclidean variable, and we can use the formula

$$\int \frac{d^d\bar{k}}{(2\pi)^d} \frac{(\bar{k}^2)^a}{(\bar{k}^2+D)^b} = \frac{\Gamma(b-a-\frac{d}{2})\Gamma(a+\frac{d}{2})}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac{d}{2})} D^{-(b-a-d/2)} \tag{2.12}$$

to evaluate it.

We use

$$\Gamma(-n+x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma + \sum_{k=1}^{n} k^{-1} + O(x)\right], \tag{2.13}$$

and

$$A^{\epsilon/2} = 1 + \frac{\epsilon}{2}\ln A + O(\epsilon^2), \tag{2.14}$$

to expand in powers of ϵ

$$\begin{aligned}
\tilde{\mu}^\epsilon \int \frac{d^d\bar{k}}{(2\pi)^d} \frac{1}{\bar{k}^2+M^2} &= \frac{1}{(4\pi)^3}\Gamma\Big(-2+\frac{\epsilon}{2}\Big)\Big(\frac{4\pi\tilde{\mu}^2}{M^2}\Big)^{\epsilon/2} \\
&= \frac{1}{2(4\pi)^3}\Big(\frac{2}{\epsilon} - \gamma + \frac{3}{2} + O(\epsilon)\Big)\Big(1 + \frac{\epsilon}{2}\ln\frac{4\pi\tilde{\mu}^2}{M^2} + O(\epsilon^2)\Big) \\
&= \frac{1}{2(4\pi)^3}\Big(\frac{2}{\epsilon} - \gamma + \frac{3}{2} + \ln\frac{4\pi\tilde{\mu}^2}{M^2} + O(\epsilon)\Big),
\end{aligned} \tag{2.15}$$

so that

$$Y = -\frac{1}{4}\frac{\varkappa}{(4\pi)^3}\Big(\frac{2}{\epsilon} - \gamma + \frac{3}{2} + \ln\frac{4\pi\tilde{\mu}^2}{M^2}\Big) + O(\varkappa^2), \tag{2.16}$$

which, in the $\epsilon \to 0$ limit, is formally infinite.

2.1.2 Corrections to the propagator

The exact scalar propagator in the Lehmann-Källén form is

$$\mathbf{\Delta}(k^2) = \frac{1}{k^2+M^2-i\epsilon} + \int_0^\infty ds\,\rho(s)\frac{1}{k^2+s-i\epsilon}, \tag{2.17}$$

where $\rho(s) \geq 0$ is called the *spectral density*. From the above, it is clear that the scalar propagator has a pole with residue 1 at $k^2 = -M^2$.

At one loop, the propagator receives the corrections from diagrams in Figure 2.2. We have

$$\frac{1}{i}\mathbf{\Delta}(k^2) = \frac{1}{i}\Delta(k^2) + \frac{1}{i}\Delta(k^2)\big[i\Pi(k^2)\big]\frac{1}{i}\Delta(k^2) + \cdots, \tag{2.18}$$

where

$$i\Pi(k^2) = \frac{1}{2}(i\varkappa)^2\Big(\frac{1}{i}\Big)^2\int\frac{d^d l}{(2\pi)^d}\Delta(l^2)\Delta((k+l)^2) - i\big(Ak^2+BM^2\big) + O(\varkappa^4), \tag{2.19}$$

where $A = Z_\phi - 1$ and $B = Z_M - 1$.

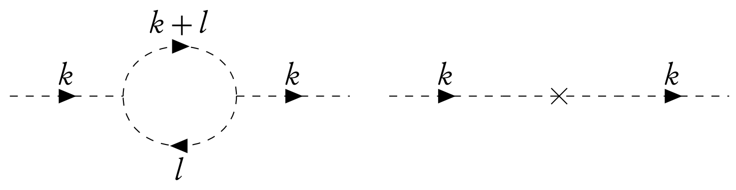


Figure 2.2: One loop correction to the scalar propagator in ϕ^3 theory.

If $i\Pi(k^2)$ is defined to be the sum of all 1PI diagrams with two external lines of momentum k, the exact propagator $\mathbf{\Delta}(k^2)$ can be written as a geometric series in $i\Pi(k^2)$,

$$\frac{1}{i}\mathbf{\Delta}(k^2) = \frac{1}{i}\Delta(k^2) + \frac{1}{i}\Delta(k^2)\big[i\Pi(k^2)\big]\frac{1}{i}\Delta(k^2) + \frac{1}{i}\Delta(k^2)\big[i\Pi(k^2)\big]\frac{1}{i}\Delta(k^2)\big[i\Pi(k^2)\big]\frac{1}{i}\Delta(k^2) + \cdots \tag{2.20}$$

so that

$$\mathbf{\Delta}(k^2) = \frac{1}{\Delta(k^2)^{-1} - \Pi(k^2)} = \frac{1}{k^2 + M^2 - i\epsilon - \Pi(k^2)}. \tag{2.21}$$

The pole of residue 1 of $\mathbf{\Delta}(k^2)$ at $k^2 = -M^2$ means that $\Pi(k^2)$ must satisfy the following

$$\Pi(-M^2) = 0 \quad \text{and} \quad \Pi'(-M^2) = 0. \tag{2.22}$$

To simplify the integrand on the RHS of (2.19), we use Feynman's trick

$$\frac{1}{A_1 \ldots A_n} = \int dF_n (x_1 A_1 + \cdots x_n A_n)^{-n}, \tag{2.23}$$

where

$$\int dF_n = (n-1)! \int_0^1 dx_1 \ldots dx_n \delta(x_1 + \cdots + x_n - 1), \tag{2.24}$$

so that $\int dF_n = 1$.

$$\begin{aligned}
\Delta(l^2)\Delta((k+l)^2) &= \frac{1}{l^2+M^2}\frac{1}{(k+l)^2+M^2} \\
&= \int_0^1 dx\big[(1-x)(l^2+M^2) + x((k+l)^2+M^2)\big]^{-2} \\
&= \int dx[(l+xk)^2 + x(1-x)k^2 + M^2]^{-2} \\
&= \int dx[q^2+D]^{-2},
\end{aligned} \tag{2.25}$$

with $q = l + xk$ and $D = x(1-x)k^2 + M^2$. We have

$$\int \frac{d^d l}{(2\pi)^d}\Delta(l^2)\Delta((k+l)^2) = \int dx \int \frac{d^d q}{(2\pi)^d}\frac{1}{(q^2+D)^2}. \tag{2.26}$$

After Wick rotating the q^0 contour and introducing the fake parameter $\tilde{\mu}$ to keep the dimensions of the integral unchanged, we can use (2.12) to do the integral,

$$
\begin{aligned}
\tilde{\mu}^{\epsilon}\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+D)^2} &= i\tilde{\mu}^{\epsilon}\int\frac{d^d\bar{q}}{(2\pi)^d}\frac{1}{(\bar{q}^2+D)^2} \\
&= \frac{i}{(4\pi)^3}\Gamma\Big(-1+\frac{\epsilon}{2}\Big)\Big(\frac{4\pi\tilde{\mu}^2}{D}\Big)^{\epsilon/2}D \\
&= -\frac{i}{(4\pi)^3}\Big(\frac{2}{\epsilon}-\gamma+1+\ln\frac{4\pi\tilde{\mu}^2}{D}+O(\epsilon)\Big)D \\
&= -\frac{i}{(4\pi)^3}\Big(\frac{2}{\epsilon}+1+\ln\frac{4\pi\tilde{\mu}}{e^{\gamma}D}+O(\epsilon)\Big)D \\
&= -\frac{i}{(4\pi)^3}\Big(\frac{2}{\epsilon}+1+\ln\frac{4\pi\tilde{\mu}}{e^{\gamma}D}+O(\epsilon)\Big)D \\
&= -\frac{i}{(4\pi)^3}\Big(\frac{2}{\epsilon}+1+\ln\frac{M^2}{D}+2\ln\frac{\mu}{M}+O(\epsilon)\Big)D, \qquad (2.27)
\end{aligned}
$$

where we have defined $\mu^2 = e^{-\gamma}4\pi\tilde{\mu}^2$, and

$$
\begin{aligned}
\Pi(k^2) &= \frac{\alpha}{2}\int dxD\ln\frac{D}{M^2}-\alpha\Big(\frac{1}{\epsilon}+1+\ln\frac{\mu}{M}\Big)\Big(\frac{1}{6}k^2+M^2\Big)-(Ak^2+BM^2)+O(\alpha^2) \\
&= \frac{\alpha}{2}\int dx\ln\frac{D}{M^2}-k^2\Big[A+\frac{\alpha}{6}\Big(\frac{1}{\epsilon}+\frac{1}{2}+\ln\frac{\mu}{M}\Big)\Big]-M^2\Big[B+\alpha\Big(\frac{1}{\epsilon}+\frac{1}{2}+\ln\frac{\mu}{M}\Big)\Big]
\end{aligned}
$$

where we have defined $\alpha = \varkappa^2/(4\pi)^3$. We choose A and B of the form

$$
A = -\frac{\alpha}{6}\Big(\frac{1}{\epsilon}+\frac{1}{2}+\ln\frac{\mu}{M}+K_A\Big), \quad B = -\alpha\Big(\frac{1}{\epsilon}+\frac{1}{2}+\ln\frac{\mu}{M}+K_B\Big), \qquad (2.28)
$$

so that the Z factors are

$$
Z_\phi = 1-\frac{\alpha}{6}\Big(\frac{1}{\epsilon}+\frac{1}{2}+\ln\frac{\mu}{M}+K_A\Big)+O(\alpha^2), \qquad (2.29)
$$

$$
Z_M = 1-\alpha\Big(\frac{1}{\epsilon}+\frac{1}{2}+\ln\frac{\mu}{M}+K_B\Big)+O(\alpha^2), \qquad (2.30)
$$

where K_A and K_B are numerical factors set by requiring $\Pi(-M^2) = 0$ and $\Pi'(-M^2) = 0$.

2.1.3 Vertex corrections

At one loop, the ϕ^3 vertex receives the following correction, due to the diagram in Figure 2.3,

$$
i\mathbf{V}_3(k_1,k_2,k_3) = iZ_\varkappa\varkappa+(i\varkappa)^3\Big(\frac{1}{i}\Big)^3\int\frac{d^6l}{(2\pi)^6}\Delta(l^2)\Delta((k_1+l)^2)\Delta((k_1+k_2+l)^2), \qquad (2.31)
$$

where $iZ_\varkappa\varkappa$ is due to the original vertex, and in writing the correction due to the loop diagram, we have taken $Z_\varkappa = 1+O(\varkappa)$ (atleast), so that the vertices contribute $i\varkappa$.

Corrections to vertices of this diagram are lumped into higher order corrections to the vertex function $\mathbf{V}_3$.

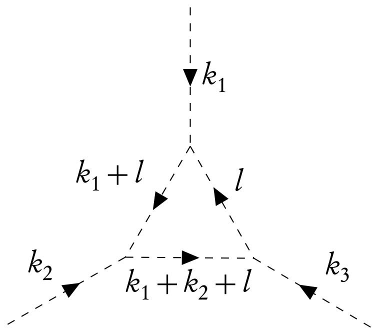


Figure 2.3: One loop correction to the ϕ^3 vertex.

In computing the integral on the RHS of (2.31), we use Feynman's trick (2.23)

$$\begin{aligned}
&\Delta(l^2)\Delta((k_1+l)^2)\Delta((k_1+k_2+l)^2)\\
&\quad=\frac{1}{l^2+M^2}\frac{1}{(k_1+l)^2+M^2}\frac{1}{(k_1+k_2+l)^2+M^2}\\
&\quad=\int dF_3\big[x_1(l^2+M^2)+x_2((k_1+l)^2+M^2)+x_3((k_1+k_2+l)^2+M^2)\big]\\
&\quad=\int dF_3\big[(l-x_1k_1+x_3k_2)^2+x_2x_1k_1^2+x_2x_3k_2^2+x_1x_3k_3^2+M^2\big]^{-3}\\
&\quad=\int dF_3\big[q^2+D\big]^{-3},
\end{aligned}\tag{2.32}$$

where $q=l-x_1k_1+x_3k_2$ and $D=x_2x_1k_1^2+x_2x_3k_2^2+x_1x_3k_3^2+M^2$. With $d^6l=d^6q$, we write the integral as

$$\int\frac{d^6l}{(2\pi)^6}\Delta(l^2)\Delta((k_1+l)^2)\Delta((k_1+k_2+l)^2)=\int dF_3\int\frac{d^6q}{(2\pi)^6}\frac{1}{(q^2+D)^3}\tag{2.33}$$

We see that this integral diverges. As before, we can isolate the divergence by analytically continuing the integral to $d=6-\epsilon$ dimensions, and then taking the $\epsilon\to 0$ limit.

After a Wick rotation, we have

$$\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+D)^3}=i\tilde{\mu}^{\epsilon}\int\frac{d^d\bar{q}}{(2\pi)^d}\frac{1}{(\bar{q}^2+D)^3},\tag{2.34}$$

where $\tilde{\mu}$ is a parameter with dimensions of mass to keep the dimensions of the integral unchanged in d dimensions. The identity (2.12) can be used to do the integral over $\bar{q}$.

$$\begin{aligned}
\mathbf{V}_3/\varkappa &= Z_\varkappa + \varkappa^2 \tilde{\mu}^\epsilon \int dF_3 \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2+D)^3} \\
&= Z_\varkappa + \frac{1}{2}\frac{\varkappa^2}{(4\pi)^3} \int dF_3\, \Gamma\Big(\frac{\epsilon}{2}\Big)\Big(\frac{4\pi\tilde{\mu}^2}{D}\Big)^{\epsilon/2} \\
&= Z_\varkappa + \frac{1}{2}\alpha \int dF_3 \Big(\frac{2}{\epsilon} - \gamma + O(\epsilon)\Big)\Big(1 + \frac{\epsilon}{2}\ln\frac{4\pi\tilde{\mu}^2}{D} + O(\epsilon^2)\Big) \\
&= Z_\varkappa + \frac{1}{2}\alpha\Big(\frac{2}{\epsilon} + \int dF_3 \ln\frac{4\pi\tilde{\mu}^2}{e^\gamma D} + O(\epsilon)\Big),
\end{aligned} \tag{2.35}$$

where we have defined $\alpha = \varkappa^2/(4\pi)^3$, and used $\int dF_3 = 1$, and (2.13). If we define $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$ and set $Z_\varkappa = 1 + C$, we have

$$\mathbf{V}_3/\varkappa = 1 + C + \alpha\Big[\frac{1}{\epsilon} + \ln\frac{\mu}{m}\Big] - \frac{1}{2}\alpha \int dF_3 \ln\frac{D}{m^2} + O(\alpha^2). \tag{2.36}$$

We can absorb the divergent $1/\epsilon$ and the unphysical constant μ into $Z_\varkappa$ by requiring

$$C = -\alpha\Big[\frac{1}{\epsilon} + \ln\frac{\mu}{M} + K_C\Big], \tag{2.37}$$

where K_C is some numerical constant, so that

$$Z_\varkappa = 1 - \alpha\Big[\frac{1}{\epsilon} + \ln\frac{\mu}{M} + K_C\Big] + O(\alpha^2), \tag{2.38}$$

where K_C is set by $\mathbf{V}_3(0,0,0) = \varkappa$.

2.1.4 Beta function

On comparing the Lagrangian with renormalized fields and parameters (in $6-\epsilon$ dimensions),

$$L = -\frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_M M^2 \phi^2 + \frac{1}{3!} Z_\varkappa \varkappa \tilde{\mu}^{\epsilon/2} \phi^3 + Y\phi, \tag{2.39}$$

and the Lagrangian with bare fields and parameters,

$$L = -\frac{1}{2}\partial^\mu \phi_0 \partial_\mu \phi_0 - \frac{1}{2} M_0^2 \phi_0^2 + \frac{1}{3!}\varkappa_0 \phi_0^3 + Y_0 \phi_0, \tag{2.40}$$

we have the following relations

$$\phi_0 = Z_\phi^{1/2}\phi \tag{2.41}$$

$$M_0 = Z_\phi^{-1/2} Z_M^{1/2} M \tag{2.42}$$

$$\varkappa_0 = Z_\phi^{-3/2} Z_\varkappa \tilde{\mu}^{\epsilon/2} \varkappa \tag{2.43}$$

$$Y_0 = Z_\phi^{-1/2} Y. \tag{2.44}$$

For $\alpha = x^2/(4\pi)^3$, we have

$$\alpha_0 = Z_\phi^{-3} Z_x^2 \tilde{\mu}^\epsilon \alpha. \tag{2.45}$$

We have computed the Z factors to one loop,

$$Z_\phi = 1 - \frac{\alpha}{6\epsilon} + O(\alpha^2), \tag{2.46}$$

$$Z_M = 1 - \frac{\alpha}{\epsilon} + O(\alpha^2), \tag{2.47}$$

$$Z_x = 1 - \frac{\alpha}{\epsilon} + O(\alpha^2). \tag{2.48}$$

To proceed with the calculation of the beta function, we define

$$K(\alpha, \epsilon) = \ln\left(Z_\phi^{-3} Z_x^2\right), \tag{2.49}$$

and note that, in general,

$$K(\alpha, \epsilon) = \sum_{n=1}^{\infty} \frac{K_n(\alpha)}{\epsilon^n}. \tag{2.50}$$

With the form of Z factors as above, we have

$$\begin{aligned} K(\alpha, \epsilon) &= \ln\left[\left(1 - \frac{\alpha}{6}\frac{1}{\epsilon}\right)^{-3}\left(1 - \alpha\frac{1}{\epsilon}\right)^2\right] \\ &= \ln\left[\left(1 + \frac{\alpha}{2}\frac{1}{\epsilon}\right)\left(1 - 2\alpha\frac{1}{\epsilon}\right)\right] \\ &= \ln\left[1 - \frac{3\alpha}{2}\frac{1}{\epsilon} + \cdots\right] \\ &= \left(-\frac{3\alpha}{2} + O(\alpha^2)\right)\frac{1}{\epsilon} + O(\epsilon^{-2}), \end{aligned} \tag{2.51}$$

so that $K_1(\alpha) = -3\alpha/2 + O(\alpha^2)$.

On physical grounds, the bare fields and parameters must be independent of the fake parameter μ. Therefore, from $\alpha_0 = Z_\phi^{-3} Z_x^2 \tilde{\mu}^\epsilon \alpha$, we have

$$\begin{aligned} 0 &= \frac{d \ln \alpha_0}{d \ln \mu} \\ &= \frac{d \ln\left(Z_\phi^{-3} Z_x^2\right)}{d \ln \mu} + \frac{d \ln \alpha}{d \ln \mu} + \epsilon \\ &= \frac{\partial K(\alpha, \epsilon)}{\partial \alpha}\frac{d\alpha}{d \ln \mu} + \frac{1}{\alpha}\frac{d\alpha}{d \ln \mu} + \epsilon \\ &= \left(\frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{K_n'(\alpha)}{\epsilon^n}\right)\frac{d\alpha}{d \ln \mu} + \epsilon, \end{aligned} \tag{2.52}$$

so that

$$\frac{d\alpha}{d \ln \mu} = -\alpha\epsilon\left(1 + \sum_{n=0}^{\infty} \frac{\alpha K_n'(\alpha)}{\epsilon^n}\right)^{-1}. \tag{2.53}$$

In a renormalizable theory, the left hand side of the above equation should be finite, and therefore, powers of $1/\epsilon$ on the right hand side must cancel. We should have,

$$\frac{d\alpha}{d\ln\mu} = -\alpha\epsilon + \beta(\alpha). \tag{2.54}$$

Comparing Equations (15) and (16), gives

$$\beta(\alpha) = \alpha^2 K_1'(\alpha) = -\frac{3\alpha^2}{2} + O(\alpha^3). \tag{2.55}$$

2.1.5 Anomalous dimension of mass

Anomalous dimension of mass is defined as

$$\gamma_M(\alpha) = \frac{1}{M}\frac{dM}{d\ln\mu}. \tag{2.56}$$

Proceeding as before, we define $A(\alpha,\epsilon) = \ln\left(Z_\phi^{-1/2} Z_M^{1/2}\right)$, and note that

$$A(\alpha,\epsilon) = \sum_{n=1}^{\infty} \frac{A_n(\alpha)}{\epsilon^n}. \tag{2.57}$$

With the Z factors as above,

$$\begin{aligned}
\ln\left(Z_\phi^{-1/2} Z_M^{1/2}\right) &= \ln\left[\left(1 - \frac{\alpha}{6}\frac{1}{\epsilon}\right)^{-1/2}\left(1 - \alpha\frac{1}{\epsilon}\right)^{1/2}\right] \\
&= \ln\left[\left(1 + \frac{\alpha}{12}\frac{1}{\epsilon}\right)\left(1 - \frac{\alpha}{2}\frac{1}{\epsilon}\right)\right] \\
&= \ln\left[1 - \frac{5\alpha}{12}\frac{1}{\epsilon}\right] \\
&= \left(-\frac{5\alpha}{12} + O(\alpha^2)\right)\frac{1}{\epsilon} + O(\epsilon^{-2}),
\end{aligned} \tag{2.58}$$

so that $A_1 = -5\alpha/12 + O(\alpha^2)$.

The bare parameter M_0 must be independent of the fake parameter μ. Therefore, from $M_0 = Z_\phi^{-1/2} Z_M^{1/2} M$, we have

$$\begin{aligned}
0 &= \frac{d\ln M_0}{d\ln\mu} \\
&= \frac{d\ln\left(Z_\phi^{-1/2} Z_M^{1/2}\right)}{d\ln\mu} + \frac{d\ln M}{d\ln\mu} \\
&= \frac{\partial A(\alpha,\epsilon)}{\partial\alpha}\frac{d\alpha}{d\ln\mu} + \frac{1}{M}\frac{dM}{d\ln\mu}.
\end{aligned} \tag{2.59}$$

$$\begin{aligned}\gamma_M(\alpha) &= -\frac{\partial A(\alpha,\epsilon)}{\partial \ln \mu}\frac{d\alpha}{d\ln \mu} \\ &= \left(\frac{A_1'(\alpha)}{\epsilon}+\frac{A_2'(\alpha)}{\epsilon^2}+\cdots\right)(\epsilon\alpha-\beta(\alpha)) \\ &= \alpha A_1'(\alpha) \,+\, \text{powers of } 1/\epsilon. \end{aligned} \tag{2.60}$$

For a renormalizable theory, γ_M should be finite and therefore the powers of $1/\epsilon$ on the right hand side must cancel. Finally, we have

$$\gamma_M(\alpha) = -\frac{5\alpha}{12}+O(\alpha^2). \tag{2.61}$$

2.1.6 Anomalous dimension of the field

Anomalous dimension of the field is defined as

$$\gamma_\phi(\alpha) = \frac{1}{2}\frac{d\ln Z_\phi}{d\ln \mu}. \tag{2.62}$$

Using the form of Z_ϕ above,

$$\ln Z_\phi = \ln\left(1-\frac{\alpha}{6\epsilon}+\cdots\right) = -\frac{\alpha}{6\epsilon}+O(\alpha^2). \tag{2.63}$$

And therefore,

$$\begin{aligned}\gamma_\phi(\alpha) &= \frac{1}{2}\frac{\partial \ln Z_\phi}{\partial \alpha}\frac{d\alpha}{d\ln\mu} \\ &= \frac{1}{2}\left(-\frac{1}{6\epsilon}+\cdots\right)(-\epsilon\alpha+\beta(\alpha)) \\ &= \frac{\alpha}{12} \,+\, \text{powers of } 1/\epsilon. \end{aligned} \tag{2.64}$$

In a renormalizable, theory, γ_ϕ should be finite and therefore the powers of $1/\epsilon$ on the right hand side must cancel. Finally,

$$\gamma_\phi(\alpha) = \frac{\alpha}{12}+O(\alpha^2). \tag{2.65}$$

2.2 Quartic interaction

Scalar field theory with a ϕ^4 self-interaction is described by the Lagrangian

$$L = -\frac{1}{2}Z_\phi \partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Z_M M^2\phi^2 - \frac{1}{4!}Z_\lambda \lambda\phi^4. \tag{2.66}$$

We notice that in d dimensions, the mass dimension of the coupling is $[\lambda] = 4-d$, and is therefore dimensionless in $d=4$ spacetime dimensions. Hence, the loop integrals in this section will be over four dimensions.

As before the Lagrangian can be organized into *free*, *interacting* and *counterterm* pieces,

$$L_0 = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}M^2\phi^2, \tag{2.67}$$

$$L_1 = -\frac{1}{4!}Z_\lambda\lambda\phi^4 + L_{\mathrm{ct}}, \tag{2.68}$$

$$L_{\mathrm{ct}} = -(Z_\phi - 1)\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - (Z_M - 1)\frac{1}{2}M^2\phi^2. \tag{2.69}$$

For the quartic interaction $Z_\lambda\lambda\phi^4$, there are no contributions to the vacuum expectation value of the field, $\langle 0|\phi(x)|0\rangle = 0$, because there are no connected diagrams with a single source. Therefore, there is no need to add a linear term to cancel tadpoles.

2.2.1 Corrections to the propagator

At the lowest order $i\Pi(k^2)$ receives corrections from diagrams of Figure 2.4,

$$i\Pi_{\phi\ \mathrm{loop}} = \frac{1}{2}(-i\lambda)\left(\frac{1}{i}\right)\int\frac{d^4l}{(2\pi)^4}\Delta(l^2). \tag{2.70}$$

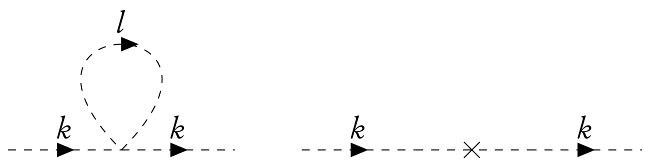


Figure 2.4: One loop correction to the scalar propagator in ϕ^4 theory.

To deal with the diverging integral above, analytically continue the integral to $d = 4 - \epsilon$ dimensions and do a Wick rotation

$$\begin{aligned}
\Pi(k^2)_{\phi\ \mathrm{loop}} &= -\frac{1}{2}\lambda\tilde{\mu}^\epsilon\int\frac{d^d\bar{l}}{(2\pi)^d}\frac{1}{\bar{l}^2 + M^2} \\
&= -\frac{1}{2}\frac{\lambda}{(4\pi)^2}\Gamma\left(-1 + \frac{\epsilon}{2}\right)\left(\frac{4\pi\tilde{\mu}^2}{M^2}\right)^{\epsilon/2}M^2 \\
&= \frac{1}{2}\frac{\lambda}{(4\pi)^2}\left(\frac{2}{\epsilon} + 1 + \ln\frac{4\pi\tilde{\mu}^2}{M^2}\right)M^2 \\
&= \frac{\lambda M^2}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{1}{2} - \ln\frac{M}{\mu}\right),
\end{aligned} \tag{2.71}$$

where $\mu^2 = e^{-\gamma}4\pi\tilde{\mu}^2$. Next, there are the counterterm contributions,

$$\Pi(k^2)_{\mathrm{ct}} = -(Z_\phi - 1)k^2 - (Z_M - 1)M^2. \tag{2.72}$$

Total contribution to $\Pi(k^2)$ at one loop,

$$\Pi(k^2) = \frac{\lambda M^2}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{1}{2} - \ln\frac{M^2}{\mu^2}\right) - (Z_\phi - 1)k^2 - (Z_M - 1)M^2. \tag{2.73}$$

We choose Z factors such that

$$Z_\phi = 1 + K_A + O(\lambda^2), \tag{2.74}$$

$$Z_M = 1 + \frac{\lambda}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{1}{2} - \frac{M^2}{\mu^2} + K_B\right) + O(\lambda^2). \tag{2.75}$$

K_A and K_B are numerical factors set by requiring $\Pi(-M^2) = 0$ and $\Pi'(-M^2) = 0$.

2.2.2 Vertex corrections

There are no three point vertices in ϕ^4 theory. The four point vertex receives the following correction at one loop due to the diagrams in Figure 2.5,

$$i\mathbf{V}_4(k_1,k_2,k_3,k_4) = -iZ_\lambda\lambda + \frac{3}{2}(-i\lambda)^2\left(\frac{1}{i}\right)^2\int\frac{d^4l}{(2\pi)^4}\Delta(l^2)\Delta((l+k_1+k_2)^2). \tag{2.76}$$

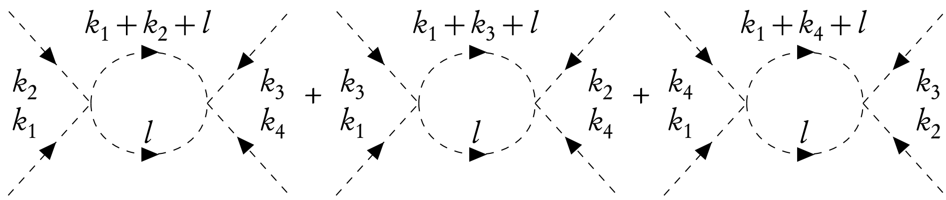


Figure 2.5: One loop corrections to the ϕ^4 vertex.

The integral above diverges. We use the usual bag of tricks.

$$\int\frac{d^4l}{(2\pi)^4}\Delta(l^2)\Delta((l+k_1+k_2)^2) \to \tilde{\mu}^\epsilon\int\frac{d^dl}{(2\pi)^d}\Delta(l^2)\Delta((l+k_1+k_2)^2) \tag{2.77}$$

$$\Delta(l^2)\Delta((l+k_1+k_2)^2) = \int dx\,\frac{1}{(q^2+D)^2}, \tag{2.78}$$

where $q = l + x(k_1+k_2)$ and $D = x(1-x)(k_1+k_2)^2 + M^2$.

$$\begin{aligned}\tilde{\mu}^\epsilon\int\frac{d^dl}{(2\pi)^d}\Delta(l^2)\Delta((l+k_1+k_2)^2) &\to i\tilde{\mu}^\epsilon\int\frac{d^d\bar{q}}{(2\pi)^d}\frac{1}{(\bar{q}^2+D)^2}\\ &= \frac{i}{(4\pi)^2}\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\epsilon/2}\\ &= \frac{2i}{(4\pi)^2}\left(\frac{1}{\epsilon} - \ln\frac{D}{M^2} + \ln\frac{\mu}{M}\right),\end{aligned} \tag{2.79}$$

where $\mu^2 = e^{-\gamma}4\pi\tilde{\mu}^2$.

$$\mathbf{V}_4/\lambda = -Z_\lambda + \frac{3\lambda}{16\pi^2}\left(-\int dx\ln\frac{D}{M^2} + \frac{1}{\epsilon} + \ln\frac{\mu}{M}\right), \tag{2.80}$$

and

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2}\left(\frac{1}{\epsilon} + \ln\frac{\mu}{M} + K_C\right), \tag{2.81}$$

where K_C is a numerical constant set by $\mathbf{V}_4(0,0,0,0) = -\lambda$.

2.2.3 Beta function

As before, we compare the Lagrangian with renormalized fields and parameters,

$$L = -\frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_M M^2 \phi^2 + \frac{1}{4!} Z_\lambda \lambda \tilde{\mu}^\epsilon \phi^4, \tag{2.82}$$

and the Lagrangian with bare fields and parameters,

$$L = -\frac{1}{2} \partial^\mu \phi_0 \partial_\mu \phi_0 - \frac{1}{2} M_0^2 \phi_0^2 + \frac{1}{4!} \lambda_0 \phi_0^4, \tag{2.83}$$

for the following relations

$$\phi_0 = Z_\phi^{1/2} \phi \tag{2.84}$$

$$M_0 = Z_\phi^{-1/2} Z_M^{1/2} M \tag{2.85}$$

$$\lambda_0 = Z_\phi^{-2} Z_\lambda \tilde{\mu}^\epsilon \lambda. \tag{2.86}$$

Computed Z factors to one loop are

$$Z_\phi = 1 + O(\lambda^2) \tag{2.87}$$

$$Z_M = 1 + \frac{\lambda}{16\pi^2} \frac{1}{\epsilon} + O(\lambda^2) \tag{2.88}$$

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2} \frac{1}{\epsilon} + O(\lambda^2). \tag{2.89}$$

As before, we define $L(\lambda, \epsilon) = \ln Z_\phi^{-2} Z_\lambda$ and note that

$$L(\lambda, \epsilon) = \sum_{n=1}^{\infty} \frac{L_n(\lambda)}{\epsilon^n}. \tag{2.90}$$

Using the form of Z factors above,

$$\ln Z_\phi^{-2} Z_\lambda = \ln\left(1 + \frac{3\lambda}{16\pi^2} \frac{1}{\epsilon}\right) = \frac{3\lambda}{16\pi^2} \frac{1}{\epsilon} + O(\lambda^2), \tag{2.91}$$

so that $L_1(\lambda) = 3\lambda/16\pi^2 + O(\lambda^2)$.

Results of the previous section hold with $\alpha \leftrightarrow \lambda$ and $K \leftrightarrow L$:

$$\frac{d\lambda}{d \ln \mu} = -\lambda\epsilon + \beta(\lambda), \tag{2.92}$$

with $\beta(\lambda) = \lambda^2 L_1'(\lambda)$ so that

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3). \tag{2.93}$$

2.2.4 Anomalous dimension of mass

Define $A(\lambda,\epsilon) = \ln Z_\phi^{-1/2} Z_M^{1/2} = \sum_{n=1}^{\infty} \epsilon^{-n} A_n(\lambda)$, and using the Z factors as above,

$$\ln\left(Z_\phi^{-1/2} Z_M^{1/2}\right) = \ln\left[\left(1+\frac{\lambda}{16\pi^2}\frac{1}{\epsilon}\right)^{1/2}\right] = \frac{\lambda}{32\pi^2}\frac{1}{\epsilon} + O(\lambda^2), \tag{2.94}$$

so that $A_1(\lambda) = \lambda/32\pi^2$. Analysis of the previous section holds, and we have

$$\gamma_M(\lambda) = \lambda A_1'(\lambda) = \frac{\lambda}{32\pi^2} + O(\lambda^2). \tag{2.95}$$

2.2.5 Anomalous dimension of the field

Anomalous dimension of the field is defined as

$$\gamma_\phi(\lambda) = \frac{1}{2}\frac{d\ln Z_\phi}{d\ln\mu}. \tag{2.96}$$

As $Z_\phi = 1 + O(\lambda^2)$, we have $\ln Z_\phi = O(\lambda^2)$ and therefore

$$\gamma_\phi(\lambda) = \frac{1}{2}\frac{\partial \ln Z_\phi}{\partial\lambda}\frac{d\lambda}{d\ln\mu}. \tag{2.97}$$

Both $\partial \ln Z_\phi/\partial\lambda$ and $d\lambda/d\ln\mu$ are $O(\lambda)$, therefore $\gamma_\phi(\lambda) = O(\lambda^2)$.

2.3 Yukawa Theory

Yukawa interaction in four spacetime dimensions is

$$L_{\text{Yukawa}} = g\phi\bar{\psi}\psi. \tag{2.98}$$

Notice that the Yukawa coupling g is dimensionless in four spacetime dimensions. However, an interaction of this kind will lead to diverging three-point and four-point scalar vertices. To absorb these divergences, we must introduce new couplings cubic and quartic in the scalar field ϕ. We also need a counterterm linear in ϕ to cancel nonvanishing tadpoles.

Renormalized Lagrangian for Yukawa theory in four spacetime dimensions is

$$L_0 = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}M^2\phi^2 \tag{2.99}$$

$$L_1 = Z_g g\phi\bar{\psi}\psi + \frac{1}{3!}Z_\varkappa \varkappa\phi^3 - \frac{1}{4!}Z_\lambda\lambda\phi^4 + L_{\text{ct}} \tag{2.100}$$

$$\begin{aligned} L_{\text{ct}} = {}& (Z_\psi - 1)i\bar{\psi}\partial\!\!\!/\psi - (Z_m - 1)m\bar{\psi}\psi \\ & - \frac{1}{2}(Z_\phi - 1)\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}(Z_M - 1)M^2\phi^2 + Y\phi. \end{aligned} \tag{2.101}$$

2.3.1 Cancelling tadpoles

Validity of the LSZ formula requires vacuum expectation values of the fields $\phi(x)$ and $\psi(x)$ be zero. While the Yukawa interaction does not result in any tadpoles involving a fermion source, there are scalar tadpoles that need to be cancelled by the counterterm $Y\phi$ (Figure 2.6).

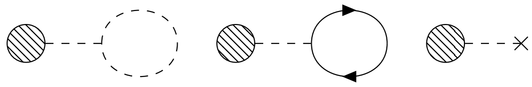


Figure 2.6: Scalar tadpoles in Yukawa theory.

We have

$$\langle 0|\phi(x)|0\rangle = \left(\frac{1}{2}(i\varkappa)\left(\frac{1}{i}\right)\Delta(0) + (-1)(ig)\left(\frac{1}{i}\right)\mathrm{Tr}\,S(0) + iY\right)\int d^4y\,\frac{1}{i}\Delta(x-y), \tag{2.102}$$

where $S(x)$ is the (position space) free fermion propagator. To cancel the tadpoles

$$iY = -\frac{1}{2}\varkappa\Delta(0) + g\,\mathrm{Tr}\,S(0), \tag{2.103}$$

where

$$\Delta(0) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+M^2-i\delta}, \tag{2.104}$$

$$\mathrm{Tr}\,S(0) = 4m\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2-i\delta}. \tag{2.105}$$

We evaluate the integrals in the (now) standard way,

$$\tilde{\mu}^{\epsilon}\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2+D} = \frac{iD}{16\pi^2}\left(\frac{2}{\epsilon} + 1 + \ln\frac{\mu^2}{D} + O(\epsilon)\right), \tag{2.106}$$

where $d = 4-\epsilon$ and $\mu^2 = e^{-\gamma}4\pi\tilde{\mu}^2$. Putting everything together

$$Y = \left(\frac{\varkappa M^2}{16\pi^2} - \frac{gm^3}{2\pi^2}\right)\left(\frac{1}{\epsilon} + \frac{1}{2} - \ln\frac{M}{\mu}\right) + \text{higher order terms}, \tag{2.107}$$

which diverges in the $\epsilon \to 0$ limit.

2.3.2 Corrections to the scalar propagator

At one loop level, $i\Pi(k^2)$ receives contributions from diagrams in Figure 2.7.

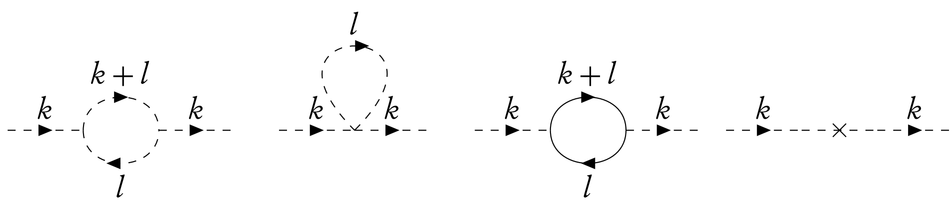


Figure 2.7: One loop corrections to the scalar propagator in Yukawa theory.

$\approx 7\pi$

$$i\Pi_{\phi^3\text{ loop}}(k^2)=\frac{1}{2}(ix)^2\left(\frac{1}{i}\right)^2\int\frac{d^4l}{(2\pi)^4}\Delta(l^2)\Delta((k+l)^2) \tag{2.108}$$

We know

$$\Delta(l^2)\Delta((k+l)^2)=\int dx\frac{1}{(q^2+D)^2}, \tag{2.109}$$

where $q=l+xk$ and $D=x(1-x)k^2+M^2$, and

$$\tilde{\mu}^{\epsilon}\int\frac{d^d\bar{q}}{(2\pi)^d}\frac{1}{(\bar{q}^2+D)^2}=\frac{1}{16\pi^2}\left(\frac{2}{\epsilon}-\ln\frac{D}{\mu^2}\right). \tag{2.110}$$

$$\begin{aligned}\Pi(k^2)_{\phi^3\text{ loop}}(k^2)&=\frac{1}{2}\tilde{\mu}^{\epsilon}x^2\int dx\int\frac{d^d\bar{q}}{(2\pi)^4}\frac{1}{(\bar{q}^2+D)^2}\\&=\frac{x^2}{2(16\pi^2)}\int dx\left(\frac{1}{\epsilon}-\ln\frac{D}{M^2}-2\ln\frac{M}{\mu}\right)\\&=\frac{x^2}{16\pi^2}\left(\frac{1}{\epsilon}-\ln\frac{M}{\mu}-\frac{1}{2}\int dx\ln\frac{D}{M^2}\right)\end{aligned} \tag{2.111}$$

Diagram with the ϕ^4 loop is identical to what was calculated in pure ϕ^4 theory,

$$\Pi(k^2)_{\phi^4\text{ loop}}(k^2)=\frac{\lambda}{16\pi^2}\left(\frac{1}{\epsilon}+\frac{1}{2}-\ln\frac{M}{\mu}\right)M^2 \tag{2.112}$$

Finally, there is the diagram with a fermion loop,

$$i\Pi_{\psi\text{ loop}}(k^2)=(-1)(ig)^2\left(\frac{1}{i}\right)^2\int\frac{d^4l}{(2\pi)^4}\operatorname{Tr}[S(\not{l})S(\not{k}+\not{l})]. \tag{2.113}$$

With

$$S(\not{k})=\frac{-\not{k}+m}{k^2+m^2-i\delta}, \tag{2.114}$$

the numerator of the integrand can be evaluated separately

$$\operatorname{Tr}[(-\not{l}+m)(-\not{k}-\not{l}+m)]=\operatorname{Tr}\left[-l(l+k)+m^2\right]=4N, \tag{2.115}$$

where $N=-l(l+k)+m^2$. As before, we simplify the denominator

$$\frac{1}{l^2+m^2}\frac{1}{(k+l)^2+m^2}=\int dx\frac{1}{(q^2+D)^2}, \tag{2.116}$$

where $q=l+xk$ and $D=x(1-x)k^2+m^2$. In terms of q, we have

$$N=-q^2+D+\text{terms linear in }q. \tag{2.117}$$

After Wick rotating and analytically continuing the integral to $d = 4-\epsilon$ dimensions,

$$\begin{aligned}\Pi_{\psi\ \text{loop}}(k^2) &= -4g^2\tilde{\mu}^{\epsilon}\int dx\int \frac{d^d\bar{q}}{(2\pi)^d}\frac{-\bar{q}^2+D}{(\bar{q}^2+D)^2}\\ &= -\frac{g^2}{4\pi^2}\Big(\frac{1}{\epsilon}+\frac{1}{6}\Big)k^2-\frac{3g^2}{2\pi^2}\Big(\frac{1}{\epsilon}+\frac{1}{6}\Big)m^2+\frac{g^2}{4\pi^2}\int dx D\ln\frac{D}{\mu^2}. \end{aligned} \tag{2.118}$$

Finally, there are the counterterms

$$\Pi_{\text{ct}}(k^2) = -(Z_\phi-1)k^2+(Z_M-1)M^2. \tag{2.119}$$

Finiteness of $\Pi(k^2)$ requires

$$Z_\phi = 1-\frac{g^2}{4\pi^2}\Big(\frac{1}{\epsilon}+\text{finite}\Big), \tag{2.120}$$

$$Z_M = 1+\Big(\frac{\varkappa^2}{16\pi^2M^2}+\frac{\lambda}{16\pi^2}-\frac{3g^2m^2}{2\pi^2M^2}\Big)\Big(\frac{1}{\epsilon}+\text{finite}\Big). \tag{2.121}$$

2.3.3 Corrections to the fermion propagator

Lehmann-Källén form of the fermion propagator,

$$\mathbf{S}(\not{k}) = \frac{-\not{k}+m}{k^2+m^2-i\delta}+\int_0^\infty ds\frac{-\not{k}\rho_1(s)+\sqrt{s}\rho_2(s)}{k^2+s-i\delta}, \tag{2.122}$$

where $\rho_1(s),\rho_2(s)\geq 0$ are spectral densities, shows that the exact propagator has a simple pole of residue 1 at $\not{k}=-m$.

If we define $i\Sigma(\not{k})$ to be the sum of all 1PI diagrams with two external fermion lines carrying momentum k, the exact propagator can be written as a geometric series,

$$\begin{aligned}\frac{1}{i}\mathbf{S}(\not{k}) &= \frac{1}{i}S(\not{k})+\frac{1}{i}S(\not{k})[i\Sigma(\not{k})]\frac{1}{i}S(\not{k})+\frac{1}{i}S(\not{k})[i\Sigma(\not{k})]\frac{1}{i}S(\not{k})[i\Sigma(\not{k})]\frac{1}{i}S(\not{k})+\cdots\\ &= S(\not{k})[1-\Sigma(\not{k})S(\not{k})]^{-1}. \end{aligned} \tag{2.123}$$

Using $S(\not{k})^{-1}=\not{k}+m$, we have the inverse of the exact propagator,

$$\mathbf{S}(\not{k})^{-1} = \not{k}+m-i\delta-\Sigma(\not{k}). \tag{2.124}$$

As before, the pole at $\not{k}=-m$ implies

$$\Sigma(-m)=0 \quad \text{and} \quad \Sigma'(-m)=0. \tag{2.125}$$

At one loop, the only correction to the fermion propagator comes from the diagrams in Figure 2.8,

$$i\Sigma_{\text{loop}}(\not{k}) = (ig)^2\Big(\frac{1}{i}\Big)^2\int\frac{d^4l}{(2\pi)^4}\Delta(l^2)S(\not{k}+\not{l}). \tag{2.126}$$

Numerator of the integrand is $N = -\not{k} - \not{l} + m$. The denominator can be simplified to

$$\frac{1}{l^2+M^2}\frac{1}{(l+k)^2+m^2} = \int dx \frac{1}{(q^2+D)^2}, \tag{2.127}$$

where $q = l + xk$ and $D = x(1-x)k^2 + m^2 + x(M^2 - m^2)$. In terms of q, the numerator becomes $N = -\not{q} - (1-x)\not{k} + m$.

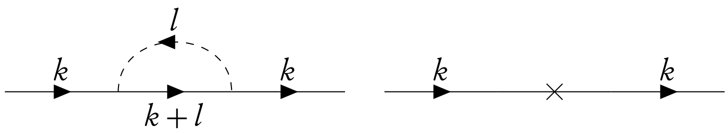


Figure 2.8: One loop corrections to the fermion propagator in Yukawa theory.

Putting all of it together, we have

$$\begin{aligned}\Sigma_{\text{loop}}(\not{k}) &= g^2 \int dx N \int \frac{d^d\bar{q}}{(2\pi)^d}\frac{\tilde{\mu}^\epsilon}{(\bar{q}^2+D)^2} \\ &= \frac{g^2}{16\pi^2}\int dx N\left(\frac{2}{\epsilon} - \ln\frac{D}{\mu}\right) \\ &= \frac{g^2}{16\pi^2}(-\not{k}+2m)\frac{1}{\epsilon} - \frac{g^2}{16\pi^2}\int dx N \ln\frac{D}{\mu^2}. \end{aligned} \tag{2.128}$$

Finally, there are the counterterm contributions

$$\Sigma_{\text{ct}}(\not{k}) = -(Z_\psi - 1)\not{k} - (Z_m - 1)m. \tag{2.129}$$

Finiteness of $\Sigma(\not{k})$ requires

$$Z_\psi = 1 - \frac{g^2}{16\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) \quad \text{and} \quad Z_m = 1 + \frac{g^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right). \tag{2.130}$$

2.3.4 Corrections to the Yukawa vertex

Contributions to the Yukawa vertex at one loop come from diagrams in Figure 2.9,

$$i\mathbf{V}_Y(k_1,k_2) = iZ_g g + (ig)^3\left(\frac{1}{i}\right)^3 \int \frac{d^4l}{(2\pi)^4}\Delta(l^2)S(\not{k}_2+\not{l})S(\not{k}_1+\not{l}) \tag{2.131}$$

$$+ (ig)^2(i\varkappa)\left(\frac{1}{i}\right)^3 \int \frac{d^4l}{(2\pi)^4} S(\not{l})\Delta((l-k_1)^2)\Delta((k_2-l)^2). \tag{2.132}$$

Note that the second integral does not diverge, and therefore will not contribute to the diverging part of Z_g. We only need to do the first integral.

We have

$$g^3\int \frac{d^4l}{(2\pi)^4}\Delta(l^2)S(\not{k}_2+\not{l})S(\not{k}_1+\not{l}) \to g^3\tilde{\mu}^\epsilon \int \frac{d^dl}{(2\pi)^d}\frac{(-\not{k}_1-\not{l}+m)(-\not{k}_2-\not{l}+m)}{(l^2+M^2)((k_1+l)^2+m^2)((k_2+l)^2+m^2)}.$$

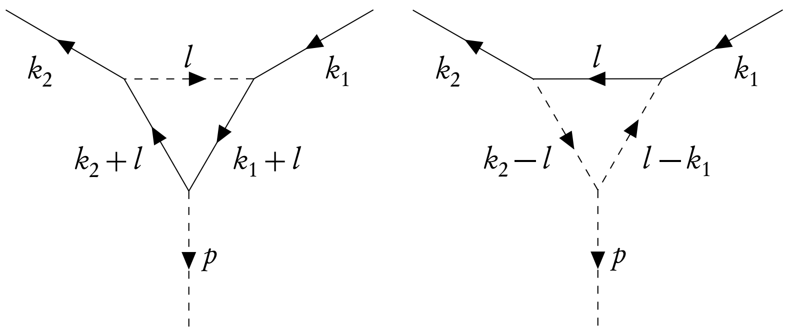


Figure 2.9: One loop corrections to the Yukawa vertex.

The denominator can be simplified,

$$\frac{(-\not{k}_1-\not{l}+m)(-\not{k}_2-\not{l}+m)}{(l^2+M^2)((k_1+l)^2+m^2)((k_2+l)^2+m^2)} = \int dF_3 \frac{1}{(q^2+D)^3}, \tag{2.133}$$

with $q = l+x_1k_1+x_2k_2$ and $D = x_1(1-x_1)k_1+x_2(1-x_2)k_2-2x_1x_2k_1k_2+x_3(M^2-m^2)$. In terms of q the numerator becomes

$$\begin{aligned} N &= -q^2+(x_1\not{k}_1+(x_2-1)\not{k}_2+m)((x_1-1)\not{k}_1+x_2\not{k}_2+m)+\text{terms linear in } q \\ &= -q^2+\tilde{N}, \end{aligned} \tag{2.134}$$

After Wick rotation we have

$$\begin{aligned} & i g^3 \tilde{\mu}^\epsilon \int dF_3 \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{-\bar{q}^2+\tilde{N}}{(\bar{q}^2+D)^3} \\ &= \frac{i g^3}{8\pi^2}\left(-\frac{1}{\epsilon}+\frac{1}{4}+\frac{1}{2}\int dF_3 \ln\frac{D}{\mu^2}+\frac{1}{4}\int dF_3 \frac{\tilde{N}}{D}\right), \end{aligned} \tag{2.135}$$

using

$$\tilde{\mu}^\epsilon \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{q}^2}{(\bar{q}^2+D)^3} = \frac{1}{16\pi^2}\left(\frac{2}{\epsilon}-\frac{1}{2}-\ln\frac{D}{\mu^2}+O(\epsilon)\right), \tag{2.136}$$

and

$$\tilde{\mu}^\epsilon \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2+D)^3} = \frac{1}{32\pi^2 D}+O(\epsilon). \tag{2.137}$$

Finally,

$$\mathbf{V}_Y(k_1,k_2)/g = Z_g - \frac{g^2}{8\pi^2}\left(\frac{1}{\epsilon}+\text{finite}\right), \tag{2.138}$$

and we can absorb the divergence in Z_g to have

$$Z_g = 1+\frac{g^2}{8\pi^2}\left(\frac{1}{\epsilon}+\text{finite}\right). \tag{2.139}$$

2.3.5 Corrections to the three point scalar vertex

At one loop the three point scalar vertex receives corrections from diagrams in Figure 2.10.

$$\begin{aligned} i\mathbf{V}_3(k_1,k_2,k_3) = iZ_\varkappa \varkappa + (i\varkappa)^3\left(\frac{1}{i}\right)^3 \int \frac{d^4 l}{(2\pi)^4}\Delta(l^2)\Delta((k_1+l)^2)\Delta((k_1+k_2+l)^2) \\ +2(-1)(ig)^3\left(\frac{1}{i}\right)^3 \int \frac{d^4 l}{(2\pi)^4}\mathrm{Tr}[S(\not{l})S(\not{k}_1+\not{l})S(\not{k}_1+\not{k}_2+\not{l})] \\ +\frac{3}{2}(-i\lambda)(i\varkappa)\left(\frac{1}{i}\right)^2 \int \frac{d^4 l}{(2\pi)^4}\Delta(l^2)\Delta((k_2+l)^2) \end{aligned} \quad (2.140)$$

Contribution from the first diagram is finite, and does not contribute to the diverging part of $Z_\varkappa$.

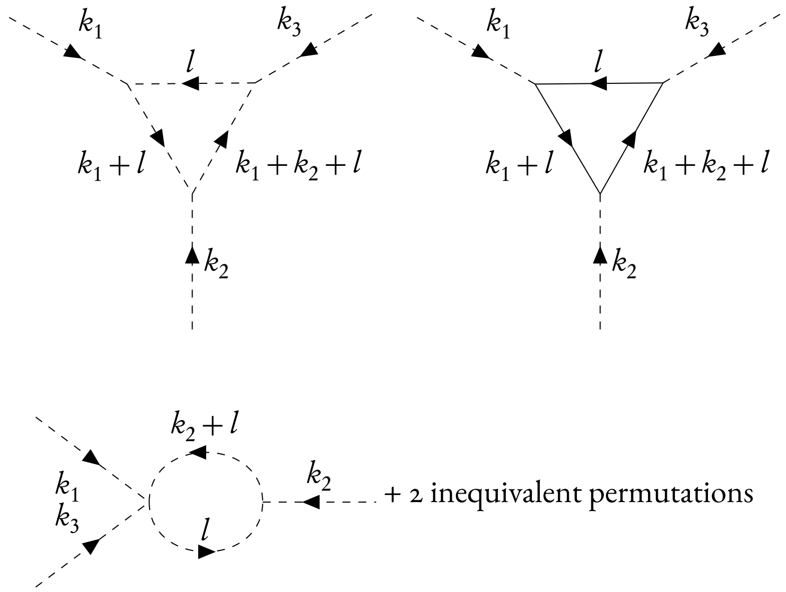


Figure 2.10: One loop corrections to the three point scalar vertex in Yukawa theory.

Let's first calculate the diagram with a fermion loop. Divergent contribution comes from the part of the numerator quadratic in l. We have

$$N \sim \mathrm{Tr}\left[(-\not{l}+m)^3\right] = -12ml^2 + \text{terms independent of } l. \quad (2.141)$$

The relevant part of the integral is

$$24img^3 \int dF_3 \int \frac{d^d\bar{l}}{(2\pi)^d}\frac{\tilde{\mu}^\epsilon \bar{l}^2}{(\bar{l}^2+D)^3} = \frac{24img^3}{16\pi^2}\left(\frac{2}{\epsilon}+\text{finite}\right) = \frac{3img^3}{\pi^2}\left(\frac{1}{\epsilon}+\text{finite}\right) \quad (2.142)$$

Next, we calculate the diagram with a ϕ^3 vertex and a ϕ^4 vertex.

$$-\frac{3i\varkappa\lambda}{2}\int dx \int \frac{d^d\bar{q}}{(2\pi)^d}\frac{\tilde{\mu}^\epsilon}{(\bar{q}^2+D)^2} = -\frac{3i\varkappa\lambda}{32\pi^2}\left(\frac{2}{\epsilon} - \int dx \ln\frac{D}{\mu^2}\right). \quad (2.143)$$

We put both these pieces together,

$$\mathbf{V}_3/x = Z_x - \left(\frac{3\lambda}{16\pi^2} - \frac{3mg^3}{\pi^2 x}\right)\left(\frac{1}{\epsilon} + \text{finite}\right), \tag{2.144}$$

so that

$$Z_x = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3mg^3}{\pi^2 x}\right)\left(\frac{1}{\epsilon} + \text{finite}\right). \tag{2.145}$$

2.3.6 Corrections to the four point scalar vertex

At one loop, the four point scalar vertex receives corrections from diagrams of Figure 2.11.

$$\begin{aligned} i\mathbf{V}_4(k_1,k_2,k_3,k_4) = & -iZ_\lambda\lambda \\ & + \frac{3}{2}(-i\lambda)^2\left(\frac{1}{i}\right)^2 \int \frac{d^4l}{(2\pi)^4} \Delta(l^2)\Delta((l+k_1+k_2)^2) \\ & + 3(ix)^4\left(\frac{1}{i}\right)^4 \int \frac{d^4l}{(2\pi)^4} \Delta(l^2)\Delta((k_1+l)^2)\Delta((k_1+k_2+l)^2)\Delta((l-k_4)^2) \\ & + 6(-1)(ig)^4\left(\frac{1}{i}\right)^4 \int \frac{d^4l}{(2\pi)^4} \text{Tr}[S(\not{l})S(\not{k}_1+\not{l})S(\not{k}_1+\not{k}_2+\not{l})S(\not{l}-\not{k}_4)] \end{aligned}$$

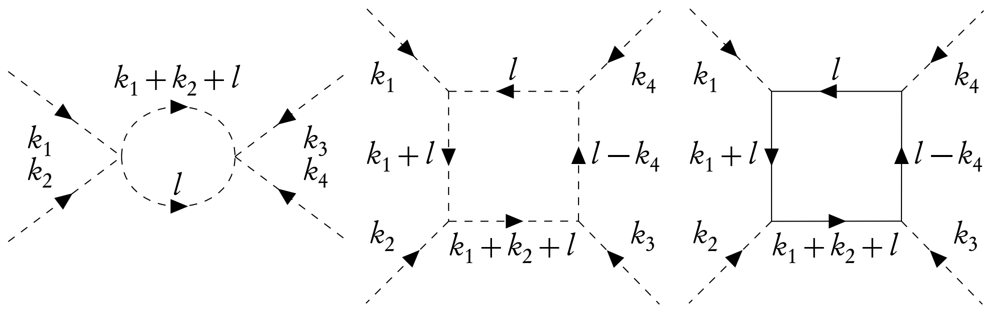


Figure 2.11: One loop corrections to the four point scalar vertex in Yukawa theory. Like in pure ϕ^4 theory, the first diagram has 2 more inequivalent permutations of external legs.

The first diagram is identical to the one in pure ϕ^4 theory and therefore

$$\mathbf{V}_{4,\phi^4\text{ loop}} = \frac{3\lambda^2}{16\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right). \tag{2.146}$$

The second diagram is finite and has no contribution to the divergent part of Z_λ. Divergent part of the first diagram comes when the numerator includes l^4:

$$N \sim (-\not{l} + m)^4 = (l^2)^2 + \text{lower order terms in } l^2. \tag{2.147}$$

We have,

$$
\begin{aligned}
i\mathbf{V}_{4,\psi\ \text{loop}} &= -24ig^4 \int \frac{d^d\bar{l}}{(2\pi)^d} \frac{\tilde{\mu}^\epsilon (\bar{l}^2)^2}{(\bar{l}^2+D)^4} \\
&= -\frac{24ig^4}{16\pi^2} \frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(4-\frac{\epsilon}{2}\right)}{\Gamma(4)\Gamma\left(2-\frac{\epsilon}{2}\right)} \left(\frac{4\pi\tilde{\mu}^\epsilon}{D}\right)^{\epsilon/2} \\
&= -\frac{3ig^4}{2\pi^2}\left(\frac{2}{\epsilon} - \frac{5}{6} - \ln\frac{D}{\mu^2} + O(\epsilon)\right). \qquad (2.148)
\end{aligned}
$$

Putting both these parts together

$$
\mathbf{V}_4/\lambda = -Z_\lambda + \left(\frac{3\lambda}{16\pi^2} - \frac{3g^4}{\pi^2\lambda}\right)\left(\frac{1}{\epsilon} + \text{finite}\right), \qquad (2.149)
$$

so that

$$
Z_\lambda = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3g^4}{\pi^2\lambda}\right)\left(\frac{1}{\epsilon} + \text{finite}\right). \qquad (2.150)
$$

2.3.7 Beta function

Comparing the Lagrangian with renormalized fields and parameters,

$$
\begin{aligned}
L = {} & iZ_\psi \bar{\psi}\partial\!\!\!/\psi - Z_m m\bar{\psi}\psi - \frac{1}{2}Z_\phi \partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Z_M M^2\phi^2 \\
& + Z_g \tilde{\mu}^{\epsilon/2} g\phi\bar{\psi}\psi + \frac{1}{3!}Z_\varkappa \tilde{\mu}^{\epsilon/2}\varkappa\phi^3 - \frac{1}{4!}Z_\lambda \tilde{\mu}^\epsilon \lambda\phi^4, \qquad (2.151)
\end{aligned}
$$

and the Lagrangian with bare field and parameters,

$$
\begin{aligned}
L = {} & i\bar{\psi}_0\partial\!\!\!/\psi_0 - m_0\bar{\psi}_0\psi_0 - \frac{1}{2}\partial^\mu\phi_0\partial_\mu\phi_0 - \frac{1}{2}M_0^2\phi_0^2 \\
& + g_0\phi_0\bar{\psi}_0\psi_0 + \frac{1}{3!}\varkappa_0\phi_0^3 - \frac{1}{4!}\lambda_0\phi_0^4, \qquad (2.152)
\end{aligned}
$$

gives the following relations

$$\psi_0 = Z_\psi^{1/2}\psi \qquad (2.153)$$

$$m_0 = Z_\psi^{-1} Z_m m \qquad (2.154)$$

$$\phi_0 = Z_\phi^{1/2}\phi \qquad (2.155)$$

$$M_0 = Z_\phi^{-1/2} Z_M^{1/2} M \qquad (2.156)$$

$$g_0 = Z_\psi^{-1} Z_\phi^{-1/2} Z_g \tilde{\mu}^{\epsilon/2} g \qquad (2.157)$$

$$\varkappa_0 = Z_\phi^{-3/2} Z_\varkappa \tilde{\mu}^{\epsilon/2}\varkappa \qquad (2.158)$$

$$\lambda_0 = Z_\phi^{-2} Z_\lambda \tilde{\mu}^\epsilon \lambda. \qquad (2.159)$$

Z factors to one loop are

$$Z_\phi = 1 - \frac{g^2}{4\pi^2}\frac{1}{\epsilon} \tag{2.160}$$

$$Z_M = 1 + \left(\frac{\varkappa^2}{16\pi^2 M^2} + \frac{\lambda}{16\pi^2} - \frac{3g^2m^2}{2\pi^2 M^2}\right)\frac{1}{\epsilon} \tag{2.161}$$

$$Z_\psi = 1 - \frac{g^2}{16\pi^2}\frac{1}{\epsilon} \tag{2.162}$$

$$Z_m = 1 + \frac{g^2}{8\pi^2}\frac{1}{\epsilon} \tag{2.163}$$

$$Z_g = 1 + \frac{g^2}{8\pi^2}\frac{1}{\epsilon} \tag{2.164}$$

$$Z_\varkappa = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3mg^3}{\pi^2\varkappa}\right)\frac{1}{\epsilon} \tag{2.165}$$

$$Z_\lambda = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3g^4}{\pi^2\lambda}\right)\frac{1}{\epsilon}. \tag{2.166}$$

To proceed with the calculation of the beta function, define

$$G(\epsilon, g, \varkappa, \lambda) = \ln\left(Z_\psi^{-1} Z_\phi^{-1/2} Z_g\right) = \sum_{n=1}^{\infty} \frac{G_n(g,\varkappa,\lambda)}{\epsilon^n}, \tag{2.167}$$

$$K(\epsilon, g, \varkappa, \lambda) = \ln\left(Z_\phi^{-3/2} Z_\varkappa\right) = \sum_{n=1}^{\infty} \frac{K_n(g,\varkappa,\lambda)}{\epsilon^n}, \tag{2.168}$$

$$L(\epsilon, g, \varkappa, \lambda) = \ln\left(Z_\phi^{-2} Z_\lambda\right) = \sum_{n=1}^{\infty} \frac{L_n(g,\varkappa,\lambda)}{\epsilon^n}, \tag{2.169}$$

and using the Z factors above, compute first order coefficients. We have

$$\begin{aligned}
\ln\left(Z_\psi^{-1} Z_\phi^{-1/2} Z_g\right) &= \ln\left[\left(1 - \frac{g^2}{16\pi^2}\frac{1}{\epsilon}\right)^{-1}\left(1 - \frac{g^2}{4\pi^2}\frac{1}{\epsilon}\right)^{-1/2}\left(1 + \frac{g^2}{8\pi^2}\frac{1}{\epsilon}\right)\right] \\
&= \ln\left[1 + \frac{5g^2}{16\pi^2}\frac{1}{\epsilon}\right] \\
&= \frac{5g^2}{16\pi^2}\frac{1}{\epsilon} + \cdots,
\end{aligned} \tag{2.170}$$

$$\begin{aligned}
\ln\left(Z_\phi^{-3/2} Z_\varkappa\right) &= \ln\left[\left(1 - \frac{g^2}{4\pi^2}\frac{1}{\epsilon}\right)^{-3/2}\left(1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3mg^3}{\pi^2\varkappa}\right)\frac{1}{\epsilon}\right)\right] \\
&= \ln\left[1 + \left(\frac{3\lambda}{16\pi^2} + \frac{3g^2}{8\pi^2} - \frac{3mg^3}{\pi^2\varkappa}\right)\frac{1}{\epsilon}\right] \\
&= \left(\frac{3\lambda}{16\pi^2} + \frac{3g^2}{8\pi^2} - \frac{3mg^3}{\pi^2\varkappa}\right)\frac{1}{\epsilon} + \cdots,
\end{aligned} \tag{2.171}$$

and

$$
\begin{aligned}
\ln\left(Z_\phi^{-2} Z_\lambda\right) &= \ln\left[\left(1-\frac{g^2}{4\pi^2}\frac{1}{\epsilon}\right)^{-2}\left(1+\left(\frac{3\lambda}{16\pi^2}-\frac{3g^4}{\pi^2\lambda}\right)\frac{1}{\epsilon}\right)\right] \\
&= \ln[1+\left(\frac{3\lambda}{16\pi^2}+\frac{g^2}{2\pi^2}-\frac{3g^4}{\pi^2\lambda}\right)\frac{1}{\epsilon}] \\
&= \left(\frac{3\lambda}{16\pi^2}+\frac{g^2}{2\pi^2}-\frac{3g^4}{\pi^2\lambda}\right)\frac{1}{\epsilon}]+\cdots
\end{aligned}
\tag{2.172}
$$

So that

$$
G_1(g,\varkappa,\lambda) = \frac{5g^2}{16\pi^2}+\cdots \tag{2.173}
$$

$$
K_1(g,\varkappa,\lambda) = \frac{3\lambda}{16\pi^2}+\frac{3g^2}{8\pi^2}-\frac{3mg^3}{\pi^2\varkappa}+\cdots \tag{2.174}
$$

$$
L_1(g,\varkappa,\lambda) = \frac{3\lambda}{16\pi^2}+\frac{g^2}{2\pi^2}-\frac{3g^4}{\pi^2\lambda}+\cdots \tag{2.175}
$$

On physical grounds, we require the bare parameters g_0, $\varkappa_0$ and λ_0 to be independent of μ. This condition leads to

$$
\sum_{n=1}^{\infty}\left(g\frac{\partial G_n}{\partial g}\frac{dg}{d\ln\mu}+g\frac{\partial G_n}{\partial \varkappa}\frac{d\varkappa}{d\ln\mu}+g\frac{\partial G_n}{\partial \lambda}\frac{d\lambda}{d\ln\mu}\right)\frac{1}{\epsilon^n}+\frac{dg}{d\ln\mu}+\frac{\epsilon g}{2}=0, \tag{2.176}
$$

$$
\sum_{n=1}^{\infty}\left(\varkappa\frac{\partial K_n}{\partial g}\frac{dg}{d\ln\mu}+\varkappa\frac{\partial K_n}{\partial \varkappa}\frac{d\varkappa}{d\ln\mu}+\varkappa\frac{\partial K_n}{\partial \lambda}\frac{d\lambda}{d\ln\mu}\right)\frac{1}{\epsilon^n}+\frac{d\varkappa}{d\ln\mu}+\frac{\epsilon\varkappa}{2}=0, \tag{2.177}
$$

$$
\sum_{n=1}^{\infty}\left(\lambda\frac{\partial L_n}{\partial g}\frac{dg}{d\ln\mu}+\lambda\frac{\partial L_n}{\partial \varkappa}\frac{d\varkappa}{d\ln\mu}+\lambda\frac{\partial L_n}{\partial \lambda}\frac{d\lambda}{d\ln\mu}\right)\frac{1}{\epsilon^n}+\frac{d\lambda}{d\ln\mu}+\epsilon\lambda=0. \tag{2.178}
$$

Requiring $dg/d\ln\mu$, $d\varkappa/d\ln\mu$ and $d\lambda/d\ln\mu$ to be finite in the $\epsilon\to 0$ limit means that we can write

$$
\frac{dg}{d\ln\mu} = -\frac{\epsilon g}{2}+\beta_g(g,\varkappa,\lambda), \tag{2.179}
$$

$$
\frac{d\varkappa}{d\ln\mu} = -\frac{\epsilon\varkappa}{2}+\beta_\varkappa(g,\varkappa,\lambda), \tag{2.180}
$$

$$
\frac{d\lambda}{d\ln\mu} = -\epsilon\lambda+\beta_\lambda(g,\varkappa,\lambda). \tag{2.181}
$$

Substituting and matching powers leads to the following expressions for the beta functions

$$
\beta_g(g,\varkappa,\lambda) = g\left(\frac{g}{2}\frac{\partial}{\partial g}+\frac{\varkappa}{2}\frac{\partial}{\partial\varkappa}+\lambda\frac{\partial}{\partial\lambda}\right)G_1, \tag{2.182}
$$

$$
\beta_\varkappa(g,\varkappa,\lambda) = \varkappa\left(\frac{g}{2}\frac{\partial}{\partial g}+\frac{\varkappa}{2}\frac{\partial}{\partial\varkappa}+\lambda\frac{\partial}{\partial\lambda}\right)K_1, \tag{2.183}
$$

$$
\beta_\lambda(g,\varkappa,\lambda) = \lambda\left(\frac{g}{2}\frac{\partial}{\partial g}+\frac{\varkappa}{2}\frac{\partial}{\partial\varkappa}+\lambda\frac{\partial}{\partial\lambda}\right)L_1. \tag{2.184}
$$

On substitution and simplification, we get

$$\beta_g(g,\varkappa,\lambda) = \frac{5g^3}{16\pi^2} + \cdots \tag{2.185}$$

$$\beta_\varkappa(g,\varkappa,\lambda) = \frac{3g^2\varkappa}{8\pi^2} - \frac{3mg^3}{\pi^2} + \frac{3\varkappa\lambda}{16\pi^2} + \cdots \tag{2.186}$$

$$\beta_\lambda(g,\varkappa,\lambda) = \frac{3\lambda^2}{16\pi^2} + \frac{g^2\lambda}{2\pi^2} - \frac{3g^4}{\pi^2} + \cdots \tag{2.187}$$

2.3.8 Anomalous dimension of mass

Define

$$A(\epsilon, g, \varkappa, \lambda) = \ln\left(Z_\phi^{-1/2} Z_M^{1/2}\right) = \sum_{n=1}^{\infty} \frac{A_n(g,\varkappa,\lambda)}{\epsilon^n} \tag{2.188}$$

and

$$B(\epsilon, g, \varkappa, \lambda) = \ln\left(Z_\psi^{-1} Z_m\right) = \sum_{n=1}^{\infty} \frac{B_n(g,\varkappa,\lambda)}{\epsilon^n}. \tag{2.189}$$

With the Z factors as above, we have

$$\begin{aligned}
\ln\left(Z_\phi^{-1/2} Z_M^{1/2}\right) &= \ln\left[\left(1 - \frac{g^2}{4\pi^2}\frac{1}{\epsilon}\right)^{-1/2}\left(1 + \left(\frac{\varkappa^2}{16\pi^2 M^2} + \frac{\lambda}{16\pi^2} - \frac{3g^2m^2}{2\pi^2M^2}\right)\frac{1}{\epsilon}\right)^{1/2}\right] \\
&= \ln\left[1 + \left(\frac{\lambda}{32\pi^2} + \frac{\varkappa^2}{32\pi^2M^2} + \frac{g^2}{8\pi^2}\left(1 - \frac{6m^2}{M^2}\right)\right)\frac{1}{\epsilon}\right] \\
&= \left[\frac{\lambda}{32\pi^2} + \frac{\varkappa^2}{32\pi^2M^2} + \frac{g^2}{8\pi^2}\left(1 - \frac{6m^2}{M^2}\right)\right]\frac{1}{\epsilon} + \cdots
\end{aligned} \tag{2.190}$$

and

$$\begin{aligned}
\ln\left(Z_\psi^{-1} Z_m\right) &= \ln\left[\left(1 - \frac{g^2}{16\pi^2}\frac{1}{\epsilon}\right)^{-1}\left(1 + \frac{g^2}{8\pi^2}\frac{1}{\epsilon}\right)\right] \\
&= \ln\left[1 + \frac{3g^2}{16\pi^2}\frac{1}{\epsilon}\right] \\
&= \frac{3g^2}{16\pi^2}\frac{1}{\epsilon} + \cdots.
\end{aligned} \tag{2.191}$$

So that

$$A_1 = \frac{\lambda}{32\pi^2} + \frac{\varkappa^2}{32\pi^2M^2} + \frac{g^2}{8\pi^2}\left(1 - \frac{6m^2}{M^2}\right) \quad \text{and} \quad B_1 = \frac{3g^2}{16\pi^2}. \tag{2.192}$$

On physical grounds, M_0 and m_0 should be independent of μ. Therefore, we must have

$$0 = \frac{d\ln M_0}{d\ln\mu} = \frac{d\ln M}{d\ln\mu} + \frac{dA}{d\ln\mu}, \tag{2.193}$$

so that

$$
\begin{aligned}
\gamma_M(g,\varkappa,\lambda) &= -\frac{dA}{d\ln\mu} \\
&= -\sum_{n=1}^{\infty}\left(\frac{dg}{d\ln\mu}\frac{\partial}{\partial g}+\frac{d\varkappa}{d\ln\mu}\frac{\partial}{\partial\varkappa}+\frac{d\lambda}{d\ln\mu}\frac{\partial}{\partial\lambda}\right)\frac{A_n(g,\varkappa,\lambda)}{\epsilon^n} \\
&= \left(\frac{g}{2}\frac{\partial}{\partial g}+\frac{\varkappa}{2}\frac{\partial}{\partial\varkappa}+\lambda\frac{\partial}{\partial\lambda}\right)A_1+\text{powers of }1/\epsilon. \qquad (2.194)
\end{aligned}
$$

Finiteness of γ_M in the $\epsilon \to 0$ limit means that the powers of $1/\epsilon$ must all cancel. Substituting A_1 in the above expression gives

$$
\gamma_M(g,\varkappa,\lambda) = \frac{g^2}{8\pi^2}\left(1-\frac{6m^2}{M^2}\right)+\frac{\varkappa}{32\pi^2M^2}+\frac{\lambda}{32\pi^2}+\cdots. \qquad (2.195)
$$

Similarly, for the fermion we have

$$
\begin{aligned}
\gamma_m(g,\varkappa,\lambda) &= \left(\frac{g}{2}\frac{\partial}{\partial g}+\frac{\varkappa}{2}\frac{\partial}{\partial\varkappa}+\lambda\frac{\partial}{\partial\lambda}\right)B_1 \\
&= \frac{3g^2}{16\pi^2}+\cdots. \qquad (2.196)
\end{aligned}
$$

2.3.9 Anomalous dimension of the field

We have

$$
\begin{aligned}
\ln Z_\phi &= \ln\left(1-\frac{g^2}{4\pi^2}\frac{1}{\epsilon}\right) \\
&= -\frac{g^2}{4\pi^2}\frac{1}{\epsilon}, \qquad (2.197)
\end{aligned}
$$

and the anomalous dimension of the scalar field is

$$
\begin{aligned}
\gamma_\phi(g,\varkappa,\lambda) &= \frac{1}{2}\frac{d\ln Z_\phi}{d\ln\mu} \\
&= \frac{1}{2}\left(\frac{\partial\ln Z_\phi}{\partial g}\frac{dg}{d\ln\mu}+\frac{\partial\ln Z_\phi}{\partial\varkappa}\frac{d\varkappa}{d\ln\mu}+\frac{\partial\ln Z_\phi}{\partial\lambda}\frac{d\lambda}{d\ln\mu}\right) \\
&= -\frac{1}{2}\frac{g}{2\pi^2}\frac{1}{\epsilon}\left(-\frac{\epsilon g}{2}+\beta_g(g,\varkappa,\lambda)\right) \\
&= \frac{g^2}{8\pi^2}+\cdots. \qquad (2.198)
\end{aligned}
$$

Similarly,

$$
\ln Z_\psi = -\frac{g^2}{16\pi^2}\frac{1}{\epsilon}, \qquad (2.199)
$$

and the anomalous dimension of the fermion field is

$$\begin{aligned}\gamma_\psi(g,\varkappa,\lambda) &= \frac{1}{2}\frac{d\ln Z_\psi}{d\ln\mu} \\ &= \frac{1}{2}\frac{\partial\ln Z_\psi}{\partial g}\frac{dg}{d\ln\mu} \\ &= -\frac{1}{2}\frac{g}{8\pi^2}\frac{1}{\epsilon}\Big(-\frac{\epsilon g}{2}+\beta_g(g,\varkappa,\lambda)\Big) \\ &= \frac{g^2}{32\pi^2}+\cdots.\end{aligned} \tag{2.200}$$

Chapter 3

Quantum Electrodynamics

Quantum electrodynamics is an abelian gauge theory based on the abelian gauge group U(1). Matter is coupled to the electromagnetic field by requiring the Lagrangian to be manifestly invariant under a local U(1) gauge transformation by replacing ordinary derivatives with a gauge covariant derivative

$$D_\mu = \partial_\mu - ieA_\mu, \tag{3.1}$$

where the gauge field $A_\mu(x)$ is a four-vector, and adding a gauge covariant kinetic term for the gauge field

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = +\frac{1}{2}A^\mu(g_{\mu\nu} - \partial_\mu\partial_\nu)A^\nu, \tag{3.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is called the electromagnetic field strength tensor.

Calculations in this chapter have been adapted from Srednicki

3.1 Coupled to spinors

For a spinor field the kinetic term is $i\bar{\psi}\partial\!\!\!/\psi$. After the replacement $\partial \to D$, we have

$$i\bar{\psi}D\!\!\!\!/\,\psi = i\bar{\psi}\partial\!\!\!/\psi + e\bar{\psi}A\!\!\!/\psi, \tag{3.3}$$

and therefore spinor electrodynamics is described by the Lagrangian

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi + e\bar{\psi}A\!\!\!/\psi. \tag{3.4}$$

After adding appropriate Z factors, it can be arranged into *free*, *interacting* and *counterterm* pieces,

$$L_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi, \tag{3.5}$$

$$L_1 = Z_1 e\bar{\psi}A\!\!\!/\psi + L_{\text{ct}}, \tag{3.6}$$

$$L_{\text{ct}} = -\frac{1}{4}(Z_3 - 1)F^{\mu\nu}F_{\mu\nu} + i(Z_2 - 1)\bar{\psi}\partial\!\!\!/\psi - (Z_m - 1)m\bar{\psi}\psi, \tag{3.7}$$

where Z_1, Z_2 and Z_3 are traditional names.

3.1.1 Tadpoles

The interacting part of the Lagrangian does not lead to any contributions to the vacuum expectation value of the fermion field: $\langle 0|\psi(x)|0\rangle = 0$, as in the case of the free theory.

There are tadpoles that contribute to the vacuum expectation value of the gauge field $A_\mu(x)$, but they all evaluate to zero. For the diagram below, the contribution is

$$\langle 0|A^\mu(x)|0\rangle \propto (-1)ie\int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}[(-\not{p}+m)\gamma^\mu]}{p^2+m^2} = -ie\int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}[-\not{p}\gamma^\mu]}{p^2+m^2} = 0, \tag{3.8}$$

because the integral is an odd function of p.

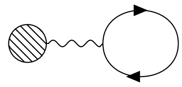


Figure 3.1: Tadpoles in in spinor electrodynamics.

3.1.2 Corrections to the photon propagator

The photon propagator receives the following loop and counterterm corrections.

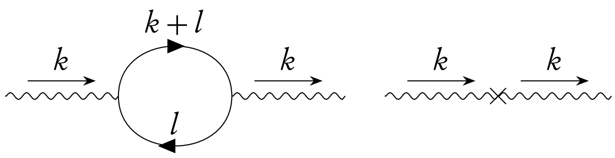


Figure 3.2: One loop correction to the photon propagator in spinor electrodynamics.

We have the loop contribution,

$$i\Pi^{\mu\nu}_{\psi\,\mathrm{loop}}(k) = -(ie)^2\left(\frac{1}{i}\right)^2\int \frac{d^4l}{(2\pi)4}\frac{\mathrm{Tr}[\gamma^\mu(-\not{l}-\not{k}+m)\gamma^\nu(-\not{l}+m)]}{((l+k)^2+m^2)(l^2+m^2)} + O(e^4) \tag{3.9}$$

and the counterterm

$$i\Pi^{\mu\nu}_{\mathrm{ct}} = -i(Z_3-1)(k^2g^{\mu\nu}-k^\mu k^\nu). \tag{3.10}$$

Simplifying the numerator of the loop contribution, we get

$$\begin{aligned}
4N^{\mu\nu} &= \mathrm{Tr}[\gamma^\mu(-\not{l}-\not{k}+m)\gamma^\nu(-\not{l}+m)]\\
&= \mathrm{Tr}[\gamma^\mu\not{l}\gamma^\nu\not{l}+\gamma^\mu\not{k}\gamma^\nu\not{l}+m\gamma^\mu\gamma^\nu]\\
&= 4(l_\alpha l_\beta+k_\alpha l_\beta)[4g^{\mu\alpha}g^{\nu\beta}-4g^{\mu\nu}g^{\alpha\beta}+4g^{\mu\beta}g^{\alpha\nu}]-4mg^{\mu\nu}\\
&= 4(2l^\mu l^\nu+k^\mu l^\nu+l^\mu k^\nu)-4g^{\mu\nu}[l(l+k)+m],
\end{aligned} \tag{3.11}$$

where in the second line we multiplied everything and dropped terms with an odd number of gamma matrices, and in the third line, traces were evaluated using

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = -4g^{\mu\nu} \tag{3.12}$$

and

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}. \tag{3.13}$$

We convert the denominator into an integral over Feynman parameters,

$$\frac{1}{[l^2+m^2][(l+k)^2+m^2]} = \int dx \frac{1}{(q^2+D)^2}, \tag{3.14}$$

where $q = l + xk$ and $D = x(1-x)k^2 + m^2$, and replace l in the numerator for q,

$$\begin{aligned} N^{\mu\nu} &= 2l^\mu l^\nu + k^\mu l^\nu + l^\mu k^\nu - g^{\mu\nu}[l(l+k)+m] \\ &= 2q^\mu q^\nu - q^2 g^{\mu\nu} - 2x(1-x)k^\mu k^\nu + x(1-x)k^2 g^{\mu\nu} - m^2 g^{\mu\nu} \\ &= \left[\left(\frac{2}{d}-1\right)q^2 + x(1-x)k^2 - m^2\right]g^{\mu\nu} - 2x(1-x)k^\mu k^\nu \\ &= 2x(1-x)\big(k^2 g^{\mu\nu} - k^\mu k^\nu\big), \end{aligned} \tag{3.15}$$

where, in the second line terms linear in q were dropped, in the third line we used

$$\int d^d q\, q^\mu q^\nu f(q^2) = \frac{g^{\mu\nu}}{d}\int d^d q\, q^2 f(q^2) \tag{3.16}$$

to make the replacement $q^\mu q^\nu \to d^{-1}q^2 g^{\mu\nu}$, and in the fourth line we used

$$\left(\frac{2}{d}-1\right)\int \frac{d^d q}{(2\pi)^d}\frac{q^2}{(q^2+D)^2} = D\int \frac{d^d q}{(2\pi)^d}\frac{1}{(q^2+D)^2}, \tag{3.17}$$

to make the replacement $(2/d-1)q^2 \to D$.

Analytically continuing to d dimensions and putting everything together, we have

$$\begin{aligned} i\Pi^{\mu\nu}_{\psi\,\mathrm{loop}} &= -4e^2\tilde{\mu}^\epsilon \int dx N^{\mu\nu}\int \frac{d^d q}{(2\pi)^d}\frac{1}{(q^2+D)^2} \\ &= -\frac{4ie^2}{8\pi^2}\frac{1}{\epsilon}\int dx\, N^{\mu\nu} + \text{finite} \\ &= -\frac{4ie^2}{8\pi^2}\frac{1}{\epsilon}(k^2 g^{\mu\nu} - k^\mu k^\nu)\int dx\, 2x(1-x) \\ &= -\frac{ie^2}{6\pi^2}\frac{1}{\epsilon}(k^2 g^{\mu\nu} - k^\mu k^\nu), \end{aligned} \tag{3.18}$$

where, in the second line we used

$$\int \frac{d^d q}{(2\pi)^d}\frac{1}{(q^2+D)^2} = \frac{i}{8\pi^2}\frac{1}{\epsilon} + \text{finite}, \tag{3.19}$$

in the third line we used the simplified form of $N^{\mu\nu}$ derived above, and in the fourth line we used $\int dx\, x(1-x) = 1/6$.

Finally, we have the photon self-energy

$$\begin{aligned}\Pi^{\mu\nu}_{1\text{ loop}} &= \Pi^{\mu\nu}_{\psi\text{ loop}} + \Pi^{\mu\nu}_{\text{ct}} \\ &= \left(-\frac{e^2}{6\pi^2}\frac{1}{\epsilon} - (Z_3 - 1)\right)(k^2 g^{\mu\nu} - k^\mu k^\nu) + \text{finite}. \end{aligned} \tag{3.20}$$

For $\Pi^{\mu\nu}_{1\text{ loop}}$ to be finite, we must have

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon}. \tag{3.21}$$

Also note that the photon self-energy is transverse, $\Pi^{\mu\nu} k_\nu = k_\mu \Pi^{\mu\nu} = 0$, as expected from gauge invariance.

3.1.3 Corrections to the fermion propagator

At one loop the fermion propagator receives the following corrections.

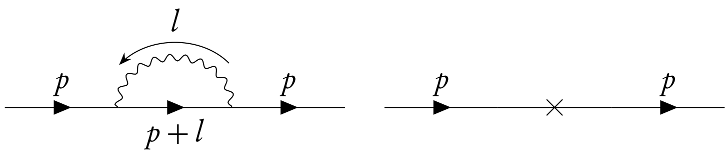


Figure 3.3: One loop correction to the fermion propagator in spinor electrodynamics

We have,

$$\begin{aligned} i\Sigma_{1\text{ loop}}(\not{p}) = (ie)^2 \left(\frac{1}{i}\right)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{\gamma^\mu(-\not{p} - \not{l} + m)\gamma_\mu}{l^2((l+p)^2 + m^2)} \\ - i(Z_2 - 1)\not{p} - i(Z_m - 1)m + O(e^4). \end{aligned} \tag{3.22}$$

As before, the denominator can be written as an integral over Feynman parameters,

$$\frac{1}{l^2((l+p)^2 + m^2)} = \int dx \frac{1}{(q^2 + D)^2}, \tag{3.23}$$

where $q = l + xp$ and $D = x(1-x)p^2 + xm^2$. The numerator can also be simplified,

$$\begin{aligned} N &= \gamma^\mu(-\not{p} - \not{l} + m)\gamma_m u \\ &= -(d-2)(\not{p} + \not{l}) - dm \\ &= -(d-2)(\not{q} + (1-x)\not{p}) - dm \\ &= -(2-\epsilon)(\not{q} + (1-x)\not{p}) - (4-\epsilon)m, \end{aligned} \tag{3.24}$$

where, to get to the second line we used $\gamma^\mu\gamma_\mu = -d$ and $\gamma^\mu \not{a}\gamma_\mu = (d-2)\not{a}$, in the third line we used $q = l + xp$ and in the fourth line we have made the substitution $d = 4-\epsilon$. Furthermore, the term linear in q integrates to zero and can be dropped.

Putting everything together, analytically continuing to $d = 4-\epsilon$ dimensions and using

$$\tilde{\mu}^\epsilon \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2+D)^2} = \frac{i}{8\pi^2}\frac{1}{\epsilon} + \text{finite}, \tag{3.25}$$

we have

$$\begin{aligned}\Sigma_{1\,\text{loop}}(\not{p}) &= -\frac{e^2}{8\pi^2}\frac{1}{\epsilon}\int dx[(2-\epsilon)(1-x)\not{p} + (4-\epsilon)m] - (Z_2-1)\not{p} - (Z_m-1)m \\ &= -\frac{e^2}{8\pi^2}\frac{1}{\epsilon}(\not{p}+4m) - (Z_2-1)\not{p} - (Z_m-1)m,\end{aligned} \tag{3.26}$$

where, in the second line we dropped all finite terms (as $\epsilon \to 0$). To keep the self-energy, $\Sigma(\not{p})$ finite, we must have

$$Z_2 = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon} \quad \text{and} \quad Z_m = 1 - \frac{e^2}{2\pi^2}\frac{1}{\epsilon}. \tag{3.27}$$

3.1.4 Corrections to the vertex

The vertex in spinor electrodynamics receives the following correction at one loop.

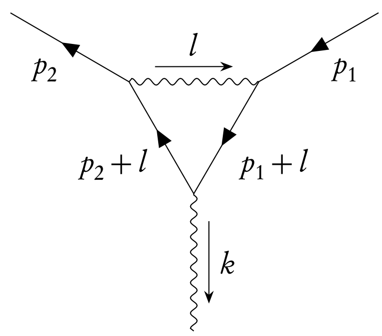


Figure 3.4: One loop correction to the spinor-spinor-photon vertex in spinor electrodynamics.

$$iV^\mu_{1\,\text{loop}} = iZ_1 e\gamma^\mu + (ie)^3\left(\frac{1}{i}\right)^3 \int \frac{d^4 l}{(2\pi)^4} \frac{\gamma^\nu(-\not{l}-\not{p}_2+m)\gamma^\mu(-\not{l}-\not{p}_1+m)\gamma_\nu}{((l+p_2)^2+m^2)((l+p_1)^2+m^2)l^2} \tag{3.28}$$

Writing the denominator as an integral over Feynman parameters, we have

$$\frac{1}{((l+p_2)^2+m^2)((l+p_1)^2+m^2)l^2} = \int dF_3 \frac{1}{(q^2+D)^3}, \tag{3.29}$$

where $q = l + x_1 p_1 + x_2 p_2$ and $D = x_1(1-x_1)p_1 + x_2(1-x_2)p_2 - 2x_1x_2p_1p_2 + (x_1 + x_2)m$.

We simplify the numerator,

$$\begin{aligned} N^\mu &= \gamma^\nu(-\not{l} - \not{p}_2 + m)\gamma^\mu(-\not{l} - \not{p}_1 + m)\gamma_\nu \\ &= \gamma^\nu(-\not{q} + x_1 p_1 - (1-x_2)p_2 + m)\gamma^\mu(-\not{q} - (1-x_1)p_1 + x_2 p_2 + m)\gamma_\nu \\ &= \gamma^\nu \not{q}\gamma^\mu \not{q}\gamma_\mu + \tilde{N}^\mu, \end{aligned} \quad (3.30)$$

where $\tilde{N}^\mu = \gamma^\nu(x_1 p_1 - (1-x_2)p_2 + m)\gamma^\mu(-(1-x_1)p_1 + x_2 p_2 + m)\gamma_\nu$, and we have dropped terms linear in q. Contribution to the divergent part of the integral comes only from the terms quadratic in q, which can be further simplified,

$$\gamma^\nu \not{q}\gamma^\mu \not{q}\gamma_\nu \to \frac{1}{d} q^2 \gamma^\nu \gamma^\alpha \gamma^\mu \gamma_\alpha \gamma_\nu = \frac{(d-2)^2}{d} q^2 \gamma^\mu, \quad (3.31)$$

where in the first line, we made the substitution $q_\alpha q_\beta \to d^{-1} q^2 g_{\alpha\beta}$, and in the second line we used $\gamma_\nu \not{a}\gamma^\nu = (d-2)\not{a}$.

Analytically continuing to d dimensions and putting all the pieces together, we have

$$\gamma^\mu \tilde{\mu}^\epsilon \int dF_3 \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{(q^2+D)^3} = \gamma^\mu \frac{i}{8\pi^2}\frac{1}{\epsilon} + \text{finite}. \quad (3.32)$$

Finally, we get

$$i\boldsymbol{V}^\mu_{\text{1 loop}} = iZ_1 e\gamma^\mu + ie\gamma^\mu\left(\frac{e^2}{8\pi^2}\frac{1}{\epsilon} + \text{finite}\right). \quad (3.33)$$

For the vertex function to be finite, we must have

$$Z_1 = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon} + O(e^4). \quad (3.34)$$

3.1.5 Beta function

Comparing the Lagrangian with renormalized parameters and fields,

$$L = -\frac{1}{4} Z_3 F^{\mu\nu} F_{\mu\nu} + iZ_2 \bar{\psi}\not{\partial}\psi - Z_m m\bar{\psi}\psi + Z_1 \tilde{\mu}^{\epsilon/2} e\bar{\psi}\not{A}\psi, \quad (3.35)$$

to the Lagrangian with bare parameters and fields,

$$L = -\frac{1}{4} F_0^{\mu\nu} F_{0\mu\nu} + i\bar{\psi}_0 \not{\partial}\psi_0 - m_0 \bar{\psi}_0 \psi_0 + e_0 \bar{\psi}_0 \not{A}_0 \psi_0, \quad (3.36)$$

we have the following relations,

$$A_0 = Z_3^{1/2} A \quad (3.37)$$

$$\psi_0 = Z_2^{1/2}\psi \quad (3.38)$$

$$m_0 = Z_2^{-1} Z_m m \quad (3.39)$$

$$e_0 = Z_3^{-1/2} Z_2^{-1} Z_1 \tilde{\mu}^{\epsilon/2} e, \quad (3.40)$$

with $\alpha = e^2/4\pi$, we also have

$$\alpha_0 = Z_3^{-1} Z_2^{-1} Z_1^2 \tilde{\mu}^\epsilon \alpha. \tag{3.41}$$

From the previous three sections, we also have the Z factors,

$$Z_1 = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon} = 1 - \frac{\alpha}{2\pi}\frac{1}{\epsilon}, \tag{3.42}$$

$$Z_2 = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon} = 1 - \frac{\alpha}{2\pi}\frac{1}{\epsilon}, \tag{3.43}$$

$$Z_m = 1 - \frac{e^2}{2\pi^2}\frac{1}{\epsilon} = 1 - \frac{2\alpha}{\pi}\frac{1}{\epsilon}, \tag{3.44}$$

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon} = 1 - \frac{2\alpha}{3\pi}\frac{1}{\epsilon}. \tag{3.45}$$

For the beta function, consider

$$\ln \alpha_0 = E + \ln \alpha + \epsilon \ln \tilde{\mu}, \tag{3.46}$$

where $E(\alpha, \epsilon) = \ln Z_3^{-1} Z_2^{-2} Z_1^2$ and due to the form of the Z factors we also have

$$E(\alpha, \epsilon) = \sum_{n=1}^{\infty} \frac{E_n(\alpha)}{\epsilon^n}. \tag{3.47}$$

By analysis of the previous chapter, we have

$$\frac{d\alpha}{d \ln \mu} = -\epsilon \alpha + \alpha^2 E_1'(\alpha), \tag{3.48}$$

and therefore, the beta function $\beta(\alpha) = \alpha^2 E_1'(\alpha)$.

Note that $Z_1^2 Z_2^{-2} = 1 + O(\alpha^2)$, so we have (at least through $O(\alpha^2)$)

$$E(\alpha, \epsilon) = -\ln Z_3 = -\ln\left(1 - \frac{2\alpha}{3\pi}\frac{1}{\epsilon}\right) = \frac{2\alpha}{3\pi}\frac{1}{\epsilon} + \cdots, \tag{3.49}$$

so that $E_1(\alpha) = 2\alpha/3\pi + O(\alpha^2)$. Finally, the beta function is

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi} + O(\alpha^3). \tag{3.50}$$

Alternatively, in terms of e,

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5). \tag{3.51}$$

This result can be easily generalized for N Dirac fields with electric charges $Q_i e$ ($i = 1, \ldots, N$). At one loop, the fermion fields and masses will be renormalized by a photon loop as above, but we must make the replacement $e \to Q_i e$. In particular,

$$Z_{2i} = 1 - \frac{Q_i^2 e^2}{8\pi^2}\frac{1}{\epsilon} \quad \text{and} \quad Z_{mi} = 1 - \frac{Q_i^2 e^2}{2\pi^2}. \tag{3.52}$$

Similarly,

$$Z_{1i} = 1 - \frac{Q_i^2 e^2}{8\pi^2}\frac{1}{\epsilon}. \tag{3.53}$$

The photon propagator, however, will be corrected by a fermion loop due to each of the N fields and we must have

$$Z_3 = 1 - \frac{\sum_{i=1}^N Q_i^2 e^2}{6\pi^2}\frac{1}{\epsilon}. \tag{3.54}$$

Proceeding as before, we note that $Z_{1i}/Z_{2i} = 1 + O(e^2)$, therefore $E(e,\epsilon) = -\ln Z_3$, and finally,

$$\beta(e) = \frac{\sum_{i=1}^N Q_i^2 e^3}{12\pi^2} + O(e^5). \tag{3.55}$$

3.1.6 Anomalous dimension of mass

As $m_0 = Z_2^{-1} Z_m m$, we have

$$\ln m_0 = A(\alpha,\epsilon) + \ln m, \tag{3.56}$$

where $A(\alpha,\epsilon) = \ln Z_2^{-1} Z_m$, and we expect $A = \sum_{n=1}^\infty A_n(\alpha)/\epsilon^n$. On physical grounds, m_0 must be independent of μ, therefore

$$0 = \frac{d\ln m_0}{d\ln\mu} = \frac{\partial A}{\partial\alpha}\frac{d\alpha}{d\ln\mu} + \frac{1}{m}\frac{dm}{d\ln\mu}. \tag{3.57}$$

Anomalous dimension of mass is defined as

$$\gamma_m(\alpha) = \frac{1}{m}\frac{dm}{d\ln\mu}, \tag{3.58}$$

therefore,

$$\begin{aligned}\gamma_m(\alpha) &= -\frac{\partial A}{\partial\alpha}\frac{d\alpha}{d\ln\mu}\\ &= -\sum_{n=1}^\infty \frac{A_n'(\alpha)}{\epsilon^n}(-\epsilon\alpha + \beta(\alpha))\\ &= \alpha A_1'(\alpha) + \text{ powers of } 1/\epsilon.\end{aligned} \tag{3.59}$$

In a renormalizable theory γ_m should be finite, so the powers of $1/\epsilon$ must all cancel.

$$A(\alpha,\epsilon) = \ln Z_2^{-1} Z_m = \left(\frac{\alpha}{2\pi} - \frac{2\alpha}{\pi}\right)\frac{1}{\epsilon} = -\frac{3\alpha}{2\pi}\frac{1}{\epsilon}, \tag{3.60}$$

so that $A_1 = -3\alpha/2\pi$ and therefore

$$\gamma_m(\alpha) = -\frac{3\alpha}{2\pi} + O(\alpha^2). \tag{3.61}$$

3.1.7 Anomalous dimension of fields

Using the definition of the anomalous dimension of fields, we have

$$\begin{aligned}\gamma_\psi(\alpha) &= \frac{1}{2}\frac{d\ln Z_2}{d\ln\mu} \\ &= \frac{1}{2}\frac{\partial \ln Z_2}{\partial\alpha}\frac{d\alpha}{d\ln\mu} \\ &= \frac{1}{2}\Big(-\frac{1}{2\pi\epsilon}+\cdots\Big)(-\alpha\epsilon+\beta(\alpha)) \\ &= \frac{\alpha}{4\pi}+\cdots. \end{aligned} \tag{3.62}$$

Similarly, for the gauge field, we have

$$\begin{aligned}\gamma_A(\alpha) &= \frac{1}{2}\frac{d\ln Z_3}{d\ln\mu} \\ &= \frac{1}{2}\frac{\partial \ln Z_3}{\partial\alpha}\frac{d\alpha}{d\ln\mu} \\ &= \frac{1}{2}\Big(-\frac{2}{3\pi\epsilon}+\cdots\Big)(-\alpha\epsilon+\beta(\alpha)) \\ &= \frac{\alpha}{3\pi}+\cdots. \end{aligned} \tag{3.63}$$

3.2 Coupled to scalars

As before we start with the manifestly gauge covariant Lagrangian for a complex scalar field with quartic self-interaction,

$$L = -(D^\mu\phi)^\dagger D_\mu\phi - M^2\phi^\dagger\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{3.64}$$

where the gauge covariant derivative, $D_\mu = \partial_\mu - ieA_\mu$, and A_μ is the abelian gauge field. The quartic interaction is needed to absorb divergences from four point scalar vertices. Expanding the covariant derivative in the kinetic term for the scalar field we have,

$$(D^\mu\phi)^\dagger D_\mu\phi = \partial^\mu\phi^\dagger\partial_\mu\phi - ieA^\mu[(\partial_\mu\phi^\dagger)^\dagger\phi - \phi^\dagger(\partial_\mu\phi)] + e^2A^\mu A_\mu\phi^\dagger\phi. \tag{3.65}$$

After introducing appropriate Z factors and organizing the Lagrangian into free, interacting and counterterm pieces, we have

$$L_0 = -\partial^\mu\phi^\dagger\partial_\mu\phi - M^2\phi^\dagger\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{3.66}$$

$$L_1 = iZ_1eA^\mu[(\partial_\mu\phi^\dagger)\phi - \phi^\dagger(\partial_\mu\phi)] - iZ_4e^2A^\mu A_\mu\phi^\dagger\phi - \frac{1}{4}Z_\lambda\lambda(\phi^\dagger\phi)^2 + L_{\mathrm{ct}}, \tag{3.67}$$

$$L_{\mathrm{ct}} = -(Z_2-1)\partial^\mu\phi^\dagger\partial_\mu\phi - (Z_M-1)M^2\phi^\dagger\phi - \frac{1}{4}(Z_3-1)F^{\mu\nu}F_{\mu\nu}, \tag{3.68}$$

so that $L = L_0 + L_1$.

3.2.1 Tadpoles

Interactions in the Lagrangian above do not give any tadpole diagrams with a scalar source. However, there are tadpoles with a photon source as below.

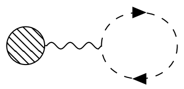


Figure 3.5: Tadpoles in scalar electrodynamics.

The diagram above is proportional to

$$\int \frac{d^4 l}{(2\pi)^4} \frac{l^\mu}{l^2 + M^2} = 0, \tag{3.69}$$

because the integrand is an odd function of the integration variable l. Hence tadpoles vanish, as anticipated by gauge invariance.

3.2.2 Corrections to the photon propagator

At one loop, the photon propagator receives corrections from the following two diagrams.

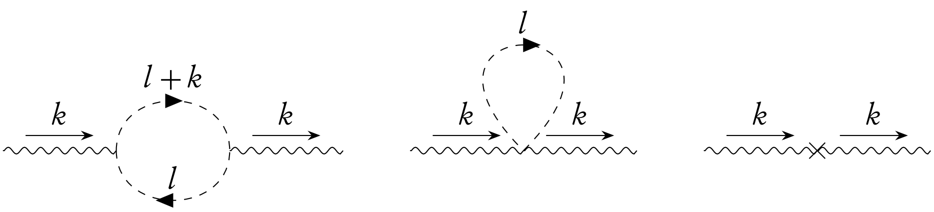


Figure 3.6: One loop corrections to the photon propagator in scalar electrodynamics.

We have the following contributions to the photon self-energy,

$$\begin{aligned} i\Pi^{\mu\nu}(k) = (iZ_1 e)^2 \left(\frac{1}{i}\right)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{(2l+k)^\mu (2l+k)^\nu}{((l+k)^2 + M^2)(l^2 + M^2)} \\ + (-2iZ_4 e^2) g^{\mu\nu} \left(\frac{1}{i}\right) \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 + M^2} \\ - i(Z_3 - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu). \end{aligned} \tag{3.70}$$

Analytically continuing to $d = 4 - \epsilon$ dimensions, replacing $e \to e\tilde{\mu}^{\epsilon/2}$, and combining the two integrals, we have

$$e^2 \tilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \frac{N^{\mu\nu}}{((l+k)^2 + M^2)(l^2 + M^2)}, \tag{3.71}$$

with

$$N^{\mu\nu} = (2l+k)^\mu (2l+k)^\nu - 2g^{\mu\nu}((l+k)^2 + M^2). \tag{3.72}$$

Writing the denominator as an integral over Feynman parameters, we have

$$\tilde{\mu}^{\epsilon}\int\frac{d^d l}{(2\pi)^d}\frac{N^{\mu\nu}}{((l+k)^2+M^2)(l^2+M^2)}=\tilde{\mu}^{\epsilon}\int dx N^{\mu\nu}\int\frac{d^d q}{(2\pi)^d}\frac{1}{(q^2+D)^2}, \quad (3.73)$$

with $q=l+xk$, $D=x(1-x)k^2+M^2$ and

$$\begin{aligned}N^{\mu\nu}&=(2l+k)^\mu(2l+k)^\nu-2g^{\mu\nu}((l+k)^2+M^2)\\&=(2q+(1-2x)k)^\mu(2q+(1-2x)k)^\nu-2g^{\mu\nu}((q+(1-x)k)^2+M^2)\\&=4q^\mu q^\nu-2g^{\mu\nu}q^2+(1-2x)^2k^\mu k^\nu-2g^{\mu\nu}[(1-x)^2k^2+M^2]\\&=2g^{\mu\nu}\Big(\frac{2}{d}-1\Big)q^2+(1-2x)^2k^\mu k^\nu-2g^{\mu\nu}[(1-x)^2k^2+M^2]\\&=2g^{\mu\nu}[x(1-x)k^2+M^2]+(1-2x)^2k^\mu k^\nu-2g^{\mu\nu}[(1-x)^2k^2+M^2]\\&=-2(1-2x)(1-x)k^2g^{\mu\nu}+(1-2x)^2k^\mu k^\nu, \end{aligned}\quad (3.74)$$

where we dropped terms linear in q, used $\int d^d q\, q^\mu q^\nu=g^{\mu\nu}d^{-1}q^2$, and

$$\Big(\frac{2}{d}-1\Big)\int\frac{d^d q}{(2\pi)^d}\frac{q^2}{(q^2+D)^2}=\int\frac{d^d q}{(2\pi)^d}\frac{D}{(q^2+D)^2}, \quad (3.75)$$

to make the replacement $(2/d-1)q^2\to D$. Putting all this together, we have

$$\begin{aligned}i\Pi^{\mu\nu}_{\text{loop}}&=e^2\int dx\, N^{\mu\nu}\int\frac{d^d q}{(2\pi)^d}\frac{\tilde{\mu}^{\epsilon}}{(q^2+D)^2}\\&=e^2\int dx\,[-2(1-2x)(1-x)k^2g^{\mu\nu}+(1-2x)^2k^\mu k^\nu]\Big(\frac{i}{8\pi^2}\frac{1}{\epsilon}+\cdots\Big)\\&=-\frac{ie^2}{24\pi^2}\frac{1}{\epsilon}(k^2g^{\mu\nu}-k^\mu k^\nu).\end{aligned}\quad (3.76)$$

With this, we have

$$\Pi^{\mu\nu}(k)=(k^2g^{\mu\nu}-k^\mu k^\nu)\Big(-\frac{e^2}{24\pi^2}+\cdots\Big)-(Z_3-1)(k^2g^{\mu\nu}-k^\mu k^\nu), \quad (3.77)$$

and to keep the self-energy finite, we have the Z factor,

$$Z_3=1-\frac{e^2}{24\pi^2}\frac{1}{\epsilon}. \quad (3.78)$$

3.2.3 Corrections to the scalar propagator

At one loop, the scalar propagator receives the following corrections.

For one loop calculations in this theory, Lorenz gauge simplifies calculations greatly. Photon propagator in the Lorenz gauge is

$$\Delta^{\mu\nu}(l)=\frac{1}{l^2-i\epsilon}\Big(g^{\mu\nu}-\frac{l^\mu l^\nu}{l^2}\Big)=\frac{P^{\mu\nu}(l)}{l^2-i\epsilon}. \quad (3.79)$$

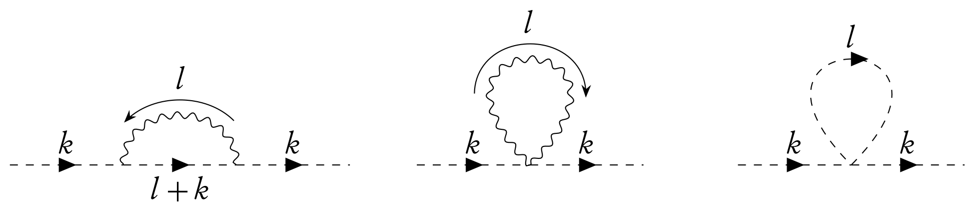


Figure 3.7: One loop corrections to the scalar propagator in scalar electrodynamics.

Note that $P^{\mu\nu}(l)l_\mu = 0$. The diagrams above yield

$$
\begin{aligned}
i\Pi(k^2) = (ie)^2\Big(\frac{1}{i}\Big)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{P^{\mu\nu}(l)(l+2k)_\mu (l+2k)_\nu}{l^2((l+k)^2+M^2)} \\
+(-2ie^2)\Big(\frac{1}{i}\Big) \int \frac{d^4 l}{(2\pi)^4} \frac{g_{\mu\nu}P^{\mu\nu}(l)}{l^2+m_\gamma^2} \\
+(-i\lambda)\Big(\frac{1}{i}\Big) \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2+M^2} - i(Z_2-1)k^2 - i(Z_M-1)M^2. \quad (3.80)
\end{aligned}
$$

The last integral above is the same as the one in pure ϕ^4 theory. For the second integral, consider $g_{\mu\nu}P^{\mu\nu} = g_{\mu\nu}(g^{\mu\nu} - l^\mu l^\nu / l^2) = d-1$, so that

$$
\begin{aligned}
\int \frac{d^4 l}{(2\pi)^4} \frac{g_{\mu\nu}P^{\mu\nu}(l)}{l^2+m_\gamma^2} &= (d-1)\int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2+m_\gamma^2} \\
&= (d-1)\int \frac{d^d l}{(2\pi)^d} \frac{\tilde{\mu}^\epsilon}{l^2+m_\gamma^2} \\
&= -(d-1)\frac{i\tilde{\mu}^\epsilon}{8\pi^2\epsilon} m_\gamma^2, \quad (3.81)
\end{aligned}
$$

which vanishes as $m_\gamma \to 0$.

For the first integral, consider the numerator

$$
\begin{aligned}
P^{\mu\nu}(l)(l+2k)_\mu (l+2k)_\nu &= 4\Big(g^{\mu\nu} - \frac{l^\mu l^\nu}{l^2}\Big)k_\mu k_\nu \\
&= \frac{4}{l^2}(l^2k^2 - (lk)^2), \quad (3.82)
\end{aligned}
$$

so that

$$
\int \frac{d^4 l}{(2\pi)^4} \frac{P^{\mu\nu}(l)(l+2k)_\mu (l+2k)_\nu}{l^2((l+k)^2+M^2)} = \int \frac{d^4 l}{(2\pi)^4} \frac{4(l^2k^2-(lk)^2)}{l^2 l^2((l+k)^2+M^2)} \quad (3.83)
$$

Writing the denominator as in integral over Feynman parameters, we have

$$
\frac{1}{l^2 l^2((l+k)^2+M^2)} = \int dF_3 \frac{1}{(q^2+D)^3}, \quad (3.84)
$$

where $q = l + x_3 k$ and $D = x_3(1-x_3)k^2 + x_3 M^2$. We also express the numerator in terms of q,

$$\begin{aligned} N &= l^2k^2 - (lk)^2 \\ &= (q - x_3 k)^2 k^2 - [(q - x_3 k)k]^2 \\ &= q^2k^2 - (qk)^2 \\ &= q^2k^2\left(1 - \frac{1}{d}\right), \end{aligned} \tag{3.85}$$

where we dropped terms linear in q and used $\int d^dq\, q^\mu q^\nu = g^{\mu\nu} d^{-1} q^2$. Since we are only interested in the divergent term, set $d = 4$. We need the following integral over q,

$$\int \frac{d^dq}{(2\pi)^d} \frac{i}{8\pi^2} \frac{1}{\epsilon} + O(\epsilon^0). \tag{3.86}$$

Putting everything together, we have

$$i\Pi(k^2) = \left(\frac{3ie^2k^2}{8\pi^2} + \frac{i\lambda M^2}{8\pi^2}\right)\frac{1}{\epsilon} - i(Z_2 - 1)k^2 - i(Z_M - 1)M^2, \tag{3.87}$$

so that

$$Z_2 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon} \quad \text{and} \quad Z_M = 1 + \frac{\lambda}{8\pi^2}\frac{1}{\epsilon}. \tag{3.88}$$

3.2.4 Corrections to the scalar-scalar-photon vertex

The scalar-scalar-photon vertex receives corrections from the following diagrams at one-loop level. External momenta cannot all be set to zero because the tree level vertex factor $iZ_1 e(k + k')_\mu$ depends on the momentum of scalar lines. In particular, both external scalars cannot have zero momentum. However, to simplify calculations as much as possible, we have made a particular choice: outgoing scalar has momentum k, while the momenta for the photon and the incoming scalar have both been set to zero.

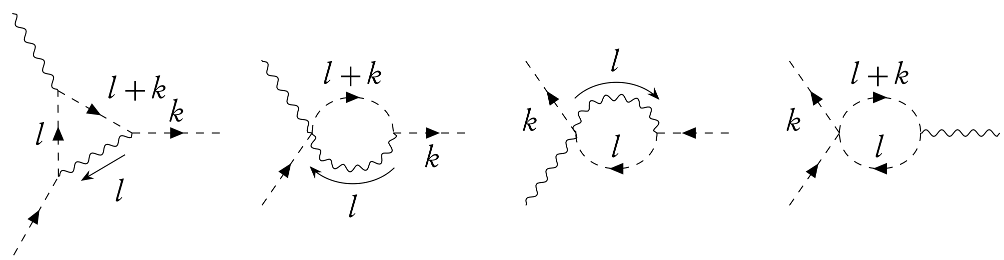


Figure 3.8: Corrections to the scalar-scalar-photon vertex in scalar electrodynamics.

Writing out the contributions explicitly, we have

$$\begin{aligned} i\boldsymbol{V}_3^{\mu} = iZ_1 ek^{\mu} &+ (ie)^3\left(\frac{1}{i}\right)^3 \int \frac{d^4 l}{(2\pi)^4} \frac{(2k+l)^{\mu}(2k+l)^{\nu} P_{\nu\rho}(l) l^{\rho}}{l^2(l^2+M^2)((l+k)^2+M^2)} \\ &+ (-2ie^2)(ie)\left(\frac{1}{i}\right)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{g^{\mu\nu} P_{\nu\rho}(l)(l+2k)^{\rho}}{l^2((l+k)^2+M^2)} \\ &+ (-2ie^2)(ie)\left(\frac{1}{i}\right)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{g^{\mu\nu} P_{\nu\rho}(l) l^{\rho}}{l^2(l^2+M^2)} \\ &+ (-i\lambda)(ie)\left(\frac{1}{i}\right)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{(2l+k)^{\mu}}{(l^2+M^2)((l+k)^2+M^2)}. \end{aligned} \tag{3.89}$$

We note that the first and third integrands are proportional to $P_{\nu\rho}(l) l^{\rho} = 0$, and therefore vanish. In the last line, after converting the denominator to an integral over Feynman parameters, we have

$$\int dx \frac{1}{(q^2+D)^2}, \tag{3.90}$$

with $q = l + xk$ and $D = x(1-x)k^2 + M^2$. The numerator becomes

$$(2l+k)^{\mu} = 2q^{\mu} + (1-2x)k^{\mu}. \tag{3.91}$$

First term vanishes after an integration over q, and the second term vanishes after integration over x. Hence, the first, third and fourth diagrams vanish.

For the second diagram, consider the numerator

$$\begin{aligned} N^{\mu} &= g^{\mu\nu} P_{\nu\rho}(l)(l+2k)_{\rho} \\ &= 2g^{\mu\nu}\left(g_{\nu\rho} - \frac{l_{\nu} l_{\rho}}{l^2}\right) k_{\rho} \\ &= \frac{2}{l^2}\left(l^2 k^{\mu} - l^{\mu}(lk)\right). \end{aligned} \tag{3.92}$$

Lumping l^{-2} with the denominator and converting the it to an integral over Feynman parameters, we will have $q = l + x_3 k$ and $D = x_3(1-x_3)k^2 + x_3 M^2$. Rewriting the numerator in terms of q, we have

$$\begin{aligned} N^{\mu} &= 2(l^2 k^{\mu} - l^{\mu}(lk)) \\ &= 2[q^2 k^{\mu} + x^2 k^2 k^{\mu} - q^{\mu}(qk) - x^2 k^{\mu} k^2] \\ &= 2q^2 k^{\mu}\left(1 - \frac{1}{d}\right), \end{aligned} \tag{3.93}$$

where we dropped terms linear in q and made the replacement $q_{\mu} q_{\nu} = g_{\mu\nu} d^{-1} q^2$. As we are interested only in the divergence, set $d = 4$. Finally, using

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^2}{(q^2+D)^3} = \frac{i}{8\pi^2} \frac{1}{\epsilon} + \cdots, \tag{3.94}$$

and putting everything together, we have

$$i\boldsymbol{V}_3^{\mu} = iZ_1ek^{\mu} - \frac{3ie^3k^{\mu}}{8\pi^2}\frac{1}{\epsilon}, \tag{3.95}$$

so that

$$Z_1 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}. \tag{3.96}$$

3.2.5 Corrections to the scalar-scalar-photon-photon vertex

The scalar-scalar-photon-photon vertex receives corrections from the following diagrams. As the tree level vertex factor $-2ieg_{\mu\nu}$ does not depend on external momenta, they have all been set to zero.

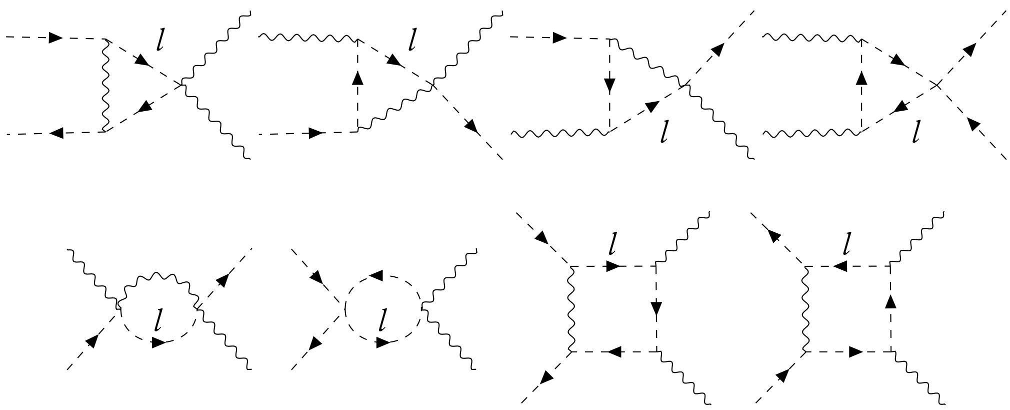


Figure 3.9: One loop corrections to the scalar-scalar-photon-photon vertex in scalar electrodynamics.

For the first diagram, the corresponding integral is

$$(-2ie^2)(ie)^2\left(\frac{1}{i}\right)^3\int\frac{d^4l}{(2\pi)^4}\frac{g^{\mu\nu}l^{\rho}l^{\sigma}P_{\rho\sigma}(l)}{l^2(l^2+M^2)(l^2+M^2)}. \tag{3.97}$$

Note that the integrand is proportional to $P_{\rho\sigma}(l)l^{\rho} = 0$, and therefore the diagram vanishes. In particular, whenever there is an external scalar attaches to an internal photon, the diagram must vanish because it would contain a term like $P_{\rho\sigma}(l)l^{\rho}$. By this argument, the first three and last two diagrams would vanish.

Contributions from the remaining three diagrams give

$$\begin{aligned} i\boldsymbol{V}_4^{\mu\nu} = &-2iZ_4e^2g_{\mu\nu} + 2(-i\lambda)(ie)^2\left(\frac{1}{i}\right)^3\int\frac{d^4l}{(2\pi)^4}\frac{(2l^{\mu})(2l^{\nu})}{(l^2+M^2)^3} \\ &+ 2(-2ie^2)^2\left(\frac{1}{i}\right)^2\int\frac{d^4l}{(2\pi)^4}\frac{g^{\mu\rho}P_{\rho\sigma}(l)g^{\sigma\nu}}{l^2(l^2+M^2)} \\ &+ (-i\lambda)(-2ie^2)\left(\frac{1}{i}\right)^2 g^{\mu\nu}\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2+M^2)^2}. \end{aligned} \tag{3.98}$$

We start with the integral in the first line. Numerator can be simplified by using the symmetric integration identity, $l^\mu l^\nu \to g^{\mu\nu} d^{-1} l^2$,

$$-2e^2\lambda g^{\mu\nu}\int \frac{d^4l}{(2\pi)^4}\frac{4d^{-1}l^2}{(l^2+M^2)^3} = -2e^2\lambda g^{\mu\nu}\left(\frac{i}{8\pi^2}\frac{1}{\epsilon}+\cdots\right), \tag{3.99}$$

where the integral was done in the usual way, and we set $d=4$, because we are only interested in the divergent piece.

For the second integral, consider the numerator

$$P^{\mu\nu}(l) = g^{\mu\nu} - \frac{l^\mu l^\nu}{l^2} = g^{\mu\nu}\left(1-\frac{1}{d}\right), \tag{3.100}$$

where we made the replacement $l^\mu l^\nu \to g^{\mu\nu} d^{-1} l^2$ because the rest of the integrand is only a function of l^2. The integrand becomes

$$8e^4\left(1-\frac{1}{d}\right)g^{\mu\nu}\int \frac{d^4l}{(2\pi)^4}\frac{1}{l^2(l^2+M^2)} \to 6e^4 g^{\mu\nu}\left(\frac{i}{8\pi^2}\frac{1}{\epsilon}+\cdots\right) = g^{\mu\nu}\frac{3ie^4}{4\pi^2}\frac{1}{\epsilon}+\cdots. \tag{3.101}$$

For the final diagram, we have

$$2e^2\lambda g^{\mu\nu}\int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2+M^2)^2} \to 2e^2\lambda g^{\mu\nu}\left(\frac{i}{8\pi^2}\frac{1}{\epsilon}+\cdots\right). \tag{3.102}$$

Putting the three pieces together, we have

$$i\boldsymbol{V}_4^{\mu\nu} = -2iZ_4e^2g^{\mu\nu} + \frac{3ie^4g^{\mu\nu}}{4\pi^2}\frac{1}{\epsilon}. \tag{3.103}$$

Absorbing the divergence in the Z factor, we have

$$Z_4 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}. \tag{3.104}$$

3.2.6 Corrections to the four scalar vertex

As before, the tree level vertex factor $-iZ_\lambda\lambda$ does not depend on external momenta, so we set them all to zero. Diagrams in which an external scalar connects to an internal photon by a three point vertex vanish for the same reason as before. Diagrams with nonzero contribution are given below.

Top two diagrams with a photon loop have identical divergent part, and both have a symmetry factor of 2. Similarly, the three diagrams below with a scalar loop have identical divergent parts and the third diagram has a symmetry factor of 2. We have

$$\begin{aligned} i\boldsymbol{V}_{4\phi} = {}& -iZ_\lambda\lambda + \left(\frac{1}{2}+\frac{1}{2}\right)(-2ie^2)^2\left(\frac{1}{i}\right)^2\int\frac{d^4l}{(2\pi)^4}\frac{g^{\mu\nu}P_{\mu\rho}(l)P_{\nu\sigma}(l)g^{\rho\sigma}}{(l^2+m_\gamma^2)^2} \\ & + \left(1+1+\frac{1}{2}\right)(-i\lambda)^2\left(\frac{1}{i}\right)^2\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2+M^2)^2}. \end{aligned} \tag{3.105}$$

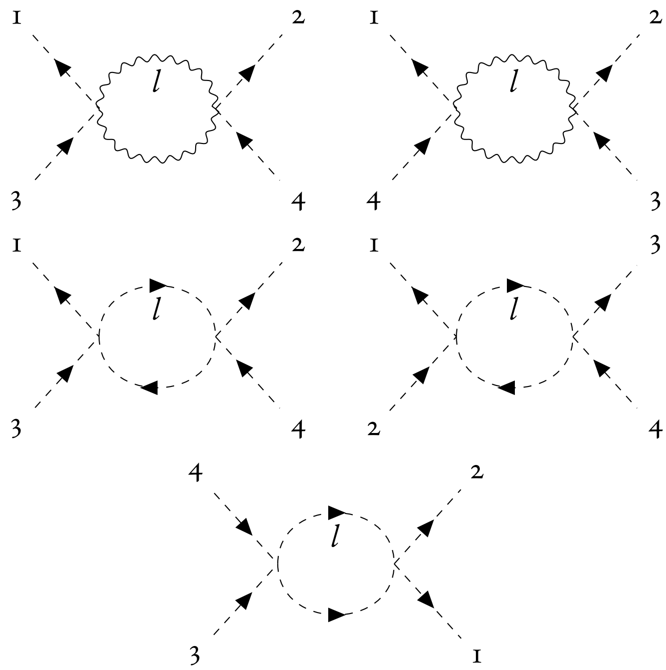


Figure 3.10: One loop corrections to the four point scalar vertex in scalar electrodynamics.

For the first integral, we use

$$g^{\mu\nu}P_{\mu\rho}(l)P_{\nu\sigma}(l)g^{\rho\sigma} = P^{\nu}_{\ \rho}P_{\nu}^{\ \rho} = P^{\nu}_{\ \nu} = (d-1), \tag{3.106}$$

and

$$\int \frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+D)^2} = \frac{i}{8\pi^2}\frac{1}{\epsilon}+\cdots, \tag{3.107}$$

to get

$$\left(\frac{1}{2}+\frac{1}{2}\right)(-2ie^2)^2\left(\frac{1}{i}\right)^2\int\frac{d^4l}{(2\pi)^4}\frac{g^{\mu\nu}P_{\mu\rho}(l)P_{\nu\sigma}(l)g^{\rho\sigma}}{(l^2+m_\gamma^2)^2} = \frac{(d-1)ie^4}{2\pi^2}\frac{1}{\epsilon}+O(\epsilon^0). \tag{3.108}$$

Similarly, for the second integral, we have

$$\left(1+1+\frac{1}{2}\right)(-i\lambda)^2\left(\frac{1}{i}\right)^2\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2+M^2)^2} = \frac{5i\lambda^2}{16\pi^2}\frac{1}{\epsilon}+O(\epsilon^0). \tag{3.109}$$

Setting $d=4$ and putting everything together,

$$\boldsymbol{V}_{4\phi} = -Z_\lambda\lambda+\left(\frac{3e^4}{2\pi^2}+\frac{5\lambda^2}{16\pi^2}\right)\frac{1}{\epsilon}+O(\epsilon^0). \tag{3.110}$$

To keep the vertex function finite, we have

$$Z_\lambda = 1+\left(\frac{3e^4}{2\pi^2\lambda}+\frac{5\lambda}{16\pi^2}\right)\frac{1}{\epsilon}. \tag{3.111}$$

3.2.7 Beta functions

Comparing the renormalized Lagrangian in $4-\epsilon$ dimensions

$$\begin{aligned} L = & -Z_2 \partial^\mu \phi^\dagger \partial_\mu \phi - Z_M M^2 \phi^\dagger \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} Z_\lambda \lambda \tilde{\mu}^\epsilon (\phi^\dagger \phi)^2 \\ & + i Z_1 e \tilde{\mu}^{\epsilon/2} A^\mu [(\partial_\mu \phi^\dagger)\phi - \phi^\dagger(\partial_\mu \phi)] - i Z_4 e^2 \tilde{\mu}^\epsilon A^\mu A_\mu \phi^\dagger \phi, \end{aligned} \tag{3.112}$$

and the Lagrangian with bare fields and parameters,

$$\begin{aligned} L = & -\partial^\mu \phi_0^\dagger \partial_\mu \phi_0 - M_0^2 \phi_0^\dagger \phi_0 - \frac{1}{4} F_0^{\mu\nu} F_{0\mu\nu} - \frac{1}{4} \lambda (\phi_0^\dagger \phi_0)^2 \\ & + i e A_0^\mu [(\partial_\mu \phi_0^\dagger)\phi_0 - \phi_0^\dagger(\partial_\mu \phi_0)] - i e^2 A_0^\mu A_{0\mu} \phi_0^\dagger \phi_0, \end{aligned} \tag{3.113}$$

we have the relations

$$\phi_0 = Z_2^{1/2} \phi \tag{3.114}$$
$$M_0 = Z_M^{1/2} Z_2^{-1/2} M \tag{3.115}$$
$$A_0^\mu = Z_3^{1/2} A^\mu \tag{3.116}$$
$$e_0 = Z_1 Z_2^{-1} Z_3^{-1/2} \tilde{\mu}^{\epsilon/2} e \tag{3.117}$$
$$e_0^2 = Z_2^{-1} Z_3^{-1} Z_4 \tilde{\mu}^\epsilon e^2 \tag{3.118}$$
$$\lambda_0 = Z_2^{-2} Z_\lambda \tilde{\mu}^\epsilon \lambda. \tag{3.119}$$

We notice that $Z_4 = Z_1^2 Z_2^{-1}$ must hold. From our computations in the four previous sections,

$$Z_1 = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\epsilon} \tag{3.120}$$
$$Z_2 = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\epsilon} \tag{3.121}$$
$$Z_3 = 1 - \frac{e^2}{24\pi^2} \frac{1}{\epsilon} \tag{3.122}$$
$$Z_4 = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\epsilon} \tag{3.123}$$
$$Z_\lambda = 1 + \left(\frac{3e^4}{2\pi^2 \lambda} + \frac{5\lambda}{16\pi^2} \right) \frac{1}{\epsilon} \tag{3.124}$$
$$Z_M = 1 + \frac{\lambda}{8\pi^2} \frac{1}{\epsilon}. \tag{3.125}$$

If we define, $E(\alpha, \epsilon) = \ln\left(Z_1^1 Z_2^{-1} Z_3^{-1/2}\right)$ and $L(\alpha, \epsilon) = \ln\left(Z_2^{-2} Z_\lambda\right)$, and notice that both can be expressed as power series in $1/\epsilon$, the beta functions are given by

$$\beta_e(e, \lambda) = e \left(\frac{e}{2} \frac{\partial}{\partial e} + \lambda \frac{\partial}{\partial \lambda} \right) E_1(e, \lambda) \tag{3.126}$$
$$\beta_\lambda(e, \lambda) = \lambda \left(\frac{e}{2} \frac{\partial}{\partial e} + \lambda \frac{\partial}{\partial \lambda} \right) L_1(e, \lambda), \tag{3.127}$$

where E_1 and L_1 are coefficients of $1/\epsilon$ in the power series for E and L respectively.

Notice that $Z_1 = Z_2$ to at least this order in e. Therefore, we have

$$E(e,\lambda) = \ln\left(Z_3^{-1}\right) = -\ln Z_3 = \left(\frac{e^2}{24\pi^2} + O(e^4)\right)\frac{1}{\epsilon} + O(\epsilon^{-2}), \tag{3.128}$$

so that $E_1 = e^2/24\pi^2 + O(e^4)$. Beta function for scalar-photon coupling is

$$\beta_e(e,\lambda) = \frac{e^3}{48\pi^2} + O(e^5). \tag{3.129}$$

Similarly, consider

$$\begin{aligned} L(e,\lambda) &= \ln\left[\left(1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}\right)^{-2}\left(1 + \left(\frac{3e^4}{2\pi^2\lambda} + \frac{5\lambda}{16\pi^2}\right)\frac{1}{\epsilon}\right)\right] \\ &= \ln\left[1 + \left(\frac{5\lambda}{16\pi^2} + \frac{-3e^2}{4\pi^2} + \frac{3e^4}{2\pi^2\lambda}\right)\frac{1}{\epsilon}\right] \\ &= \left(\frac{5\lambda}{16\pi^2} + \frac{-3e^2}{4\pi^2} + \frac{3e^4}{2\pi^2\lambda}\right)\frac{1}{\epsilon} \end{aligned} \tag{3.130}$$

so that

$$L_1(e,\lambda) = \frac{5\lambda}{16\pi^2} + \frac{-3e^2}{4\pi^2} + \frac{3e^4}{2\pi^2\lambda} \tag{3.131}$$

and the beta function

$$\beta_\lambda(\alpha,\lambda) = \frac{5\lambda^2}{16\pi^2} - \frac{3e^2\lambda}{4\pi^2} + \frac{3e^4}{2\pi^2}. \tag{3.132}$$

3.2.8 Anomalous dimensions

For anomalous dimension of the electromagnetic field, proceeding as before

$$\gamma_A(e,\lambda) = \frac{1}{2}\frac{d\ln Z_3}{d\ln\mu} = \frac{e^2}{48\pi^2}. \tag{3.133}$$

Similarly, for anomalous dimension of scalar field, we have

$$\gamma_\phi(e,\lambda) = \frac{1}{2}\frac{d\ln Z_2}{d\ln\mu} = -\frac{3e^2}{16\pi^2}. \tag{3.134}$$

Finally, for anomalous dimension of mass, we need

$$B(e,\lambda) = \ln\left(Z_M^{1/2} Z_2^{-1/2}\right) = \left(\frac{\lambda}{16\pi^2} - \frac{3e^2}{16\pi^2}\right)\frac{1}{\epsilon} + O(\epsilon^{-2}) \tag{3.135}$$

and therefore

$$\gamma_M(e,\lambda) = \left(\frac{e}{2}\frac{\partial}{\partial e} + \lambda\frac{\partial}{\partial\lambda}\right)B_1(e,\lambda) = \frac{\lambda}{16\pi^2} - \frac{3e^2}{16\pi^2}. \tag{3.136}$$

Chapter 4

Nonabelian Gauge Theory

In the minimal prescription of electrodynamics, matter is coupled to the gauge field by requiring the Lagrangian to be manifestly invariant under a local gauge transformation. Consider a set of N fields $\phi_j(x)$ in an N-dimensional representation R of the gauge group. A local gauge transformation is given by

$$\phi_j(x) \to \exp\left[-i g \Gamma^a(x) T_R^a\right]_j^k \phi_k(x), \tag{4.1}$$

where g is a dimensionless constant, and T_R^a are generators of the group in its N-dimensional representation R. Indices j, k which correspond to a representation of the gauge group are called color indices.

A Lagrangian that is invariant under a global transformation of the gauge group can be made to respect the local symmetry by replacing the ordinary derivative by a gauge covariant derivative

$$D_\mu = \partial_\mu - i g A_\mu^a T_R^a, \tag{4.2}$$

where $A_\mu^a(x)$ are a set of gauge fields, and color indices are supressed. A gauge invariant Lagrangian for just the gauge field, also called a Yang-Mills Lagrangian is given by

$$L_{\mathrm{YM}} = -\frac{1}{4} G^{e\mu\nu} G^e_{\mu\nu}, \tag{4.3}$$

where $G^e_{\mu\nu} = \partial_\mu A^e_\nu - \partial_\nu A^e_\mu$, is the field strength tensor for the gauge field $A^e_\mu(x)$. With $G^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu$ (where f^{abc} are structure factors for the gauge group), kinetic term for the gauge field can be expanded,

$$\begin{aligned} -\frac{1}{4} G^{e\mu\nu} G^e_{\mu\nu} = &-\frac{1}{2}\partial^\mu A^{e\nu} \partial_\mu A^e_\nu + \frac{1}{2}\partial^\mu A^{e\nu} \partial_\nu A^e_\mu \\ &- g f^{abe} A^{a\mu} A^{b\nu} \partial_\mu A^{e\nu} - \frac{1}{4} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu . \end{aligned} \tag{4.4}$$

By doing an integration-by-parts on the first two terms and throwing away the surface integral, we have

$$\begin{aligned} -\frac{1}{4} G^{e\mu\nu} G^e_{\mu\nu} = &+\frac{1}{2} A^{e\mu} (g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu) A^{e\nu} \\ &- g f^{abe} A^{a\mu} A^{b\nu} \partial_\mu A^{e\nu} - \frac{1}{4} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu . \end{aligned} \tag{4.5}$$

To quantize the gauge field, we need to gauge-fix by adding ghosts

$$\begin{aligned} L_{\text{gh}} &= -\partial^\mu \bar{c}^b D^{bc}_\mu c^c \\ &= -\partial^\mu \bar{c}^c \partial_\mu c^c + g f^{abc} A^a_\mu \partial^\mu \bar{c}^b c^c, \end{aligned} \tag{4.6}$$

and a gauge fixing term

$$L_{\text{gf}} = -\frac{1}{2}\xi^{-1}\partial^\mu A^c_\mu \partial^\nu A^c_\nu = +\frac{1}{2}\xi^{-1} A^{c\mu}\partial_\mu\partial_\nu A^{c\nu}, \tag{4.7}$$

where for the second equality we have done an integration-by-parts. Putting everything together, we have a quantum variant of the Yang-Mills Lagrangian

$$\begin{aligned} L_{\text{YM}} + L_{\text{gh}} + L_{\text{gf}} = &\frac{1}{2}A^{e\mu}(g_{\mu\nu} - \partial_\mu\partial_\nu)A^{e\nu} + \frac{1}{2}\xi^{-1}A^{e\mu}\partial_\mu\partial_\nu A^{e\nu} - \partial^\mu \bar{c}^a \partial_\mu c^a \\ &- g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^c_\nu - \frac{1}{4} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu \\ &+ g f^{abc} A^c_\mu \partial^\mu \bar{c}^a c^b. \end{aligned} \tag{4.8}$$

Calculations leading up to the calculation of the beta function in nonabelian gauge theory with spinors has been adapted from Srednicki.

4.1 Coupled to spinors

As before coupling to spinors occurs by replacing the ordinary derivative with the partial derivative.

$$L_{\text{fermion}} = i\bar{\psi}^i_I \not{D}_i^{\,j} \psi_{jI} - m_I \bar{\psi}^i_I \psi_{iI}, \tag{4.9}$$

where I is the *flavour index*. In quantum chromodynamics, the gauge group is SU(3) and quarks are in its fundamental representation. As a result, each quark comes in three colours. There are six flavours, and each flavour has a different mass.

Expanding out the kinetic term, we have

$$i\bar{\psi}^i_I \not{D}_i^{\,j} \psi_{jI} = i\bar{\psi}^i_I \not{\partial} \psi_{iI} + g A^a_\mu (T^a_R)_i^{\,j} \bar{\psi}^i_I \gamma^\mu \psi_{jI}, \tag{4.10}$$

where the index on a runs from 1 to $D(A)$, the index on i, j runs from 1 to $D(R)$ and the index on I runs from 1 to n_F[1]. In quantum chromodynamics, $D(A) = 8$, $D(R) = 3$ and $n_F = 6$.

[1] $D(R)$ is the dimension of the representation R of the gauge group, A stands for its adjoint representation, and n_F is the number of fermions in the theory

Putting everything together and adding appropriate renormalizing Z factors for loop calculations, we have

$$L_0 = +\frac{1}{2}A^{a\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{a\nu} + \frac{1}{2}\xi^{-1}A^{a\mu}\partial_\mu\partial_\nu A^{a\nu} - \partial^\mu\bar{c}^a\partial_\mu c^a + i\bar{\psi}^i_I\not{\partial}\psi_{iI} - Z_{mI}m_I\bar{\psi}^i_I\psi_{iI} \tag{4.11}$$

$$L_1 = -Z_{3g}gf^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \frac{1}{4}Z_{4g}g^2f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu + Z_{1'}gf^{abc}A^a_\mu\partial^\mu\bar{c}^c c^b + Z_{1I}gA^a_\mu(T^a_R)_i^{\ j}\bar{\psi}^i_I\gamma^\mu\psi_{jI} + L_{\text{ct}}, \tag{4.12}$$

$$L_{\text{ct}} = +\frac{1}{2}(Z_3 - 1)A^{a\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{a\nu} - (Z_{2'} - 1)\partial^\mu\bar{c}^a\partial_\mu c^a + i(Z_{2I} - 1)\bar{\psi}_{iI}\not{\partial}\psi_{iI} - (Z_{mI} - 1)m_I\bar{\psi}_{iI}\psi_{iI}. \tag{4.13}$$

4.1.1 Corrections to the gluon propagator

At one loop, the gluon propagator is corrected by the following five diagrams.

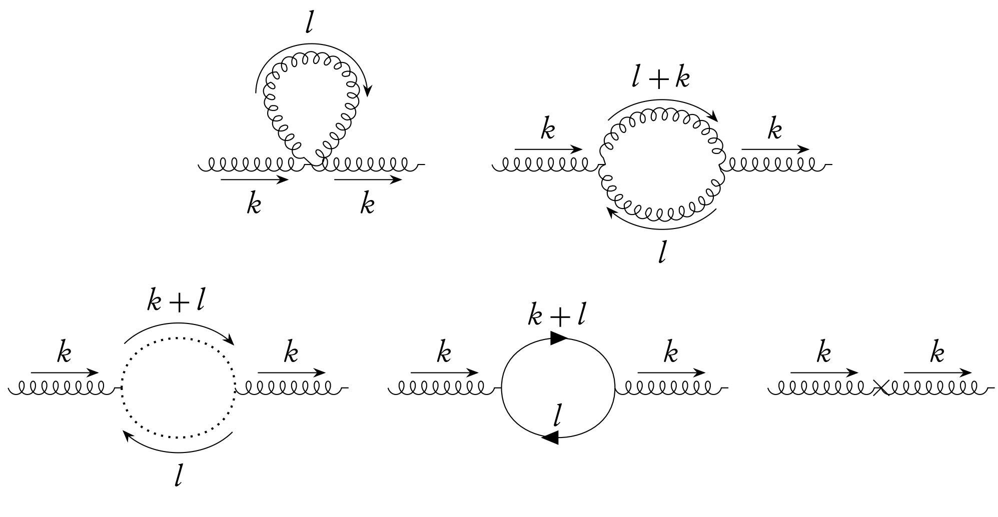


Figure 4.1: One loop corrections to the gluon propagator in spinor gauge theory.

The first diagram is proportional to $\int d^4l/l^2$. However, this integral vanishes after dimensional regularization.

$$\int \frac{d^4l}{(2\pi)^4}\frac{1}{l^2} \to \tilde{\mu}^\epsilon \int \frac{d^dl}{(2\pi)^d}\frac{1}{l^2 + m_g^2}, \tag{4.14}$$

where we have introduced an infrared cutoff m_g, for "gluon mass".

$$\begin{aligned}\tilde{\mu}^{\epsilon}\int\frac{d^d l}{(2\pi)^d}\frac{1}{l^2+m_g^2}&=\frac{i\tilde{\mu}^{\epsilon}}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)m_g^{-2(1-d/2)}\\&=\frac{i m_g^2}{(4\pi)^2}\Gamma\left(-1+\frac{\epsilon}{2}\right)\left(\frac{4\pi\tilde{\mu}^2}{m_g^2}\right)^{\epsilon/2}\\&=\frac{-i m_g^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma+1\right)\left(1+\frac{\epsilon}{2}\ln\left(\frac{4\pi\tilde{\mu}^2}{m_g^2}\right)\right)\\&=\frac{-i m_g^2}{16\pi^2}\left(\frac{2}{\epsilon}+1-\ln\frac{m_g^2}{\mu^2}\right),\end{aligned}\tag{4.15}$$

which vanishes as $m_g \to 0$.

For the second diagram, we have

$$\begin{aligned}&i\Pi^{ab}_{\mu\nu,\text{ gluon loop}}(k)\\&=\frac{1}{2}\left(\frac{1}{i}\right)^2\int\frac{d^4 l}{(2\pi)^4}\frac{g^{\alpha\sigma}g^{\beta\rho}i\mathbf{V}^{bcd}_{\nu\rho\sigma}(k,-k-l,l)i\mathbf{V}^{adc\sigma\rho}_{\mu\alpha\beta}(-k,-l,k+l)}{l^2(l+k)^2},\end{aligned}\tag{4.16}$$

where

$$i\mathbf{V}^{abc}_{\mu\nu\rho}(p,q,r)=gf^{abc}[(q-r)_{\mu}g_{\nu\rho}+(r-p)_{\nu}g_{\rho\mu}+(p-q)_{\rho}g_{\mu\nu}]\tag{4.17}$$

is the factor associated with the three gluon vertex. We can expand the numerator,

$$\begin{aligned}&g^{\alpha\sigma}g^{\beta\rho}i\mathbf{V}^{bcd}_{\nu\rho\sigma}(k,-k-l,l)i\mathbf{V}^{adc}_{\mu\alpha\beta}(-k,-l,k+l)\\&\quad=g^2f^{bcd}f^{adc}[(-k-2l)_{\nu}g_{\rho\sigma}+(-k+l)_{\rho}g_{\nu\sigma}+(2k+l)_{\sigma}g_{\nu\rho}]\\&\qquad[(-k-2l)_{\mu}g^{\sigma\rho}+(2k+l)^{\sigma}\delta^{\rho}_{\mu}+(-k+l)^{\rho}\delta^{\sigma}_{\mu}].\end{aligned}\tag{4.18}$$

Colour factors simplify as

$$\begin{aligned}f^{bcd}f^{adc}&=-(-if^{bcd})(-if^{adc})\\&=-(T^b_A)^{cd}(T^a_A)^{dc}\\&=-\operatorname{Tr}\left(T^bT^a\right)\\&=-T(A)\delta^{ab},\end{aligned}\tag{4.19}$$

and we write

$$g^{\alpha\sigma}g^{\beta\rho}i\mathbf{V}^{bcd}_{\nu\rho\sigma}(k,-k-l,l)i\mathbf{V}^{adc}_{\mu\alpha\beta}(-k,-l,k+l)=-g^2T(A)\delta^{ab}N_{\mu\nu},\tag{4.20}$$

where

$$\begin{aligned}N_{\mu\nu}=&[(-k-2l)_{\nu}g_{\rho\sigma}+(-k+l)_{\rho}g_{\nu\sigma}+(2k+l)_{\sigma}g_{\nu\rho}]\\&\times[(-k-2l)_{\mu}g^{\sigma\rho}+(2k+l)^{\sigma}\delta^{\rho}_{\mu}+(-k+l)^{\rho}\delta^{\sigma}_{\mu}].\end{aligned}\tag{4.21}$$

We express the denominator as an integral over Feynman parameters,

$$\frac{1}{l^2(l+k)^2} = \int dx \frac{1}{(q^2+D)^2}, \tag{4.22}$$

where $q = l + xk$ and $D = x(1-x)k^2$, and write $N_{\mu\nu}$ in terms of q,

$$\begin{aligned}
N_{\mu\nu} &= [(2q+(1-2x)k)_\nu g_{\rho\sigma} + (-q+(1+x)k)_\rho g_{\nu\sigma} + (-q-(2-x)k)_\sigma g_{\nu\rho}] \\
&\quad [(2q+(1-2x)k)_\mu g^{\rho\sigma} + (-q-(2-x)k)^\sigma \delta^\rho_\mu + (-q+(1+x)k)^\rho \delta^\sigma_\mu] \\
&= 4q_\mu q_\nu g_{\rho\sigma} g^{\rho\sigma} - 2q_\nu q^\sigma g_{\rho\sigma}\delta^\rho_\mu - 2q_\nu q^\rho g_{\rho\sigma}\delta^\sigma_\mu - 2q_\rho q_\mu g_{\nu\sigma} g^{\rho\sigma} + q_\rho q^\sigma g_{\nu\sigma}\delta^\rho_\mu \\
&\quad + q_\rho q^\rho g_{\nu\sigma}\delta^\sigma_\mu - 2q_\sigma q_\mu g_{\nu\rho} g^{\rho\sigma} + q_\sigma q^\sigma g_{\nu\rho}\delta^\rho_\mu + q_\sigma q^\rho g_{\nu\rho}\delta^\sigma_\mu \\
&\quad + (1-2x)^2 k_\mu k_\nu g_{\rho\sigma} g^{\rho\sigma} - (1-2x)(2-x)k_\nu k^\sigma g_{\rho\sigma}\delta^\rho_\mu \\
&\quad + (1-2x)(1+x)k_\nu k^\rho g_{\rho\sigma}\delta^\sigma_\mu + (1-2x)(1+x)k_\rho k_\mu g^{\rho\sigma} g_{\nu\sigma} \\
&\quad - (1+x)(2-x)k_\rho k^\sigma g_{\nu\sigma}\delta^\rho_\mu + (1+x)^2 k_\rho k^\rho g_{\nu\sigma}\delta^\sigma_\mu \\
&\quad - (2-x)(1-2x)k_\sigma k_\mu g_{\nu\rho} g^{\rho\sigma} + (2-x)^2 k_\sigma k^\sigma g_{\nu\rho}\delta^\rho_\mu \\
&\quad - (2-x)(1+x)k_\sigma k^\rho g_{\nu\rho}\delta^\sigma_\mu \ + \text{ terms linear in } q \qquad (4.23) \\
&= 4d q_\mu q_\nu - 2q_\mu q_\nu - 2q_\mu q_\nu - 2q_\mu q_\nu + q_\mu q_\nu + q^2 g_{\mu\nu} - 2q_\mu q_\nu + q^2 g_{\mu\nu} + q_\mu q_\nu \\
&\quad + (4x^2-4x+1)d k_\mu k_\nu - (2x^2-5x+2)k_\mu k_\nu + (-2x^2-x+1)k_\mu k_\nu \\
&\quad + (-2x^2-x+1)k_\mu k_\nu - (-x^2+x+2)k_\mu k_\nu + (x^2+2x+1)k^2 g_{\mu\nu} \\
&\quad - (2x^2-5x+2)k_\mu k_\nu + (x^2-4x+4)k^2 g_{\mu\nu} - (-x^2+x+2)k_\mu k_\nu \qquad (4.24) \\
&= (4d-6)q_\mu q_\nu + 2q^2 g_{\mu\nu} + ((4d-6)x^2 - (4d-6)x - (6-d))k_\mu k_\nu \\
&\quad + (2x^2-2x+5)k^2 g_{\mu\nu}. \qquad (4.25)
\end{aligned}$$

To simplify, we dropped terms linear in q and used $g_{\mu\nu}g^{\mu\nu} = d$. Moreover, we can make the replacements $q_\mu q_\nu \to d^{-1} g_{\mu\nu} q^2$ and $q^2 \to (2/d-1)^{-1} D$ with $D = x(1-x)k^2$, such that

$$\begin{aligned}
N_{\mu\nu} &= 10q_\mu q_\nu + 2q^2 g_{\mu\nu} + (2x^2-2x+5)k^2 g_{\mu\nu} + (10x^2-10x-2)k_\mu k_\nu \\
&\to \frac{9}{2} q^2 g_{\mu\nu} + (2x^2-2x+5)k^2 g_{\mu\nu} + (10x^2-10x-2)k_\mu k_\nu \\
&\to (11x^2-11x+5)k^2 g_{\mu\nu} + (10x^2-10x-2)k_\mu k_\nu.
\end{aligned} \tag{4.26}$$

As we are only interested in the divergent part of the integral, we put $d = 4$ throughout. Putting everything together, we have

$$\begin{aligned}
i\Pi^{ab}_{\mu\nu,\text{ gluon loop}}(k) &= \frac{1}{2} g^2 T(A)\delta^{ab} \int dx N_{\mu\nu} \int \frac{d^d q}{(2\pi)^d} \frac{\tilde{\mu}^\epsilon}{(q^2+D)^2} \\
&= \frac{1}{2} g^2 T(A)\delta^{ab} \left(\frac{19}{6} k^2 g_{\mu\nu} - \frac{11}{3} k_\mu k_\nu\right)\left(\frac{i}{8\pi^2}\frac{1}{\epsilon} + \cdots\right) \\
&\quad \frac{i g^2}{16\pi^2} T(A)\delta^{ab} \left(\frac{19}{6} k^2 g_{\mu\nu} - \frac{11}{3} k_\mu k_\nu\right)\left(\frac{1}{\epsilon} + \cdots\right)
\end{aligned} \tag{4.27}$$

For the third diagram, we have

$$i\Pi^{ab}_{\mu\nu,\,\text{ghost loop}}(k) = (-1)\left(\frac{1}{i}\right)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{i\mathbf{V}^{acd}_{\mu}(l+k,l)\,i\mathbf{V}^{bdc}_{\nu}(l,l+k)}{l^2(l+k)^2}, \tag{4.28}$$

where

$$i\mathbf{V}^{abc}_{\mu}(q,r) = g f^{abc} q_{\mu}, \tag{4.29}$$

is the factor associated with the ghost-gluon-ghost vertex. We expand the numerator,

$$i\mathbf{V}^{acd}_{\mu}(l+k,l)\,i\mathbf{V}^{bdc}_{\nu}(l,l+k) = g^2 f^{acd} f^{bdc} (l+k)_{\mu} l_{\nu}. \tag{4.30}$$

Simplify the colour factor by

$$\begin{aligned} f^{acd} f^{bdc} &= -(-if^{acd})(-if^{bdc}) \\ &= -(T^a_A)^{cd}(T^b_A)^{dc} \\ &= -\operatorname{Tr} T^a_A T^b_A \\ &= -T(A)\delta^{ab}, \end{aligned} \tag{4.31}$$

so that

$$i\mathbf{V}^{acd}_{\mu}(l+k,l)\,i\mathbf{V}^{bdc}_{\nu}(l,l+k) = -g^2 T(A)\delta^{ab} N_{\mu\nu}, \tag{4.32}$$

where $N_{\mu\nu} = (l+k)_{\mu} l_{\nu}$. Write the denominator as an integral over Feynman parameters,

$$\frac{1}{l^2(l+k)^2} = \int dx \frac{1}{(q^2+D)^2}, \tag{4.33}$$

where $q = l + xk$ and $D = x(1-x)k^2$, and write the numerator in terms of q,

$$N_{\mu\nu} = q_{\mu} q_{\nu} - x(1-x) k_{\mu} k_{\nu} + \text{terms linear in } q. \tag{4.34}$$

Make replacements $q_{\mu} q_{\nu} \to d^{-1} q^2 g_{\mu\nu}$, $q^2 \to (2/d-1)^{-1} D$ and drop terms linear in q to get

$$N_{\mu\nu} \to -\frac{1}{2} x(1-x) k^2 g_{\mu\nu} - x(1-x) k_{\mu} k_{\nu}. \tag{4.35}$$

Putting everything together,

$$\begin{aligned} i\Pi^{ab}_{\mu\nu,\,\text{ghost loop}}(k) &= -g^2 T(A)\delta^{ab} \int dx N_{\mu\nu} \int \frac{d^d q}{(2\pi)^d} \frac{\tilde{\mu}^{\epsilon}}{(q^2+D)} \\ &= -g^2 T(A)\delta^{ab} \left(-\frac{1}{12} k^2 g_{\mu\nu} - \frac{1}{6} k_{\mu} k_{\nu}\right)\left(\frac{i}{8\pi^2}\frac{1}{\epsilon} + \cdots\right) \\ &= \frac{ig^2}{8\pi^2} T(A)\delta^{ab} \left(\frac{1}{12} k^2 g_{\mu\nu} + \frac{1}{6} k_{\mu} k_{\nu}\right)\left(\frac{1}{\epsilon} + \cdots\right). \end{aligned} \tag{4.36}$$

For the fourth diagram, we have

$$
\begin{aligned}
&i\Pi^{ab}_{\mu\nu,\text{ fermion loop}}(k)\\
&\quad=(-1)(ig)^2\left(\frac{1}{i}\right)^2\operatorname{Tr}\left(T^b_R T^a_R\right)\int\frac{d^4l}{(2\pi)^4}\frac{\operatorname{Tr}\left(\gamma_\nu(-\not{l}-\not{k}+m)\gamma_\mu(-\not{l}+m)\right)}{(l^2+m^2)((l+k)^2+m^2)}.
\end{aligned}
$$

Other than the colour factor $\operatorname{Tr}\left(T^b_R T^a_R\right)=T(R)\delta^{ab}$, the expression is identical to the similar electrodynamics diagram. Moreover, if the theory has n_F families of fermions, then each of the fermions will contribute with a loop. Each of these contributions will be identical. As a result, we have

$$
i\Pi^{ab}_{\mu\nu,\text{ fermion loop}}(k)=-\frac{ig^2}{6\pi^2}T(R)n_F\delta^{ab}\left(k^2g_{\mu\nu}-k_\mu k_\nu\right)\left(\frac{1}{\epsilon}+\cdots\right). \tag{4.37}
$$

Counterterm contribution to the gluon propagator is

$$
\Pi^{ab}_{\mu\nu,\text{ ct}}=-(Z_3-1)(k^2g_{\mu\nu}-k_\mu k_\nu)\delta^{ab}=\delta^{ab}(k^2g_{\mu\nu}-k_\mu k_\nu)\Pi_{\text{ct}}(k^2) \tag{4.38}
$$

Summing all the contributions, we have

$$
\begin{aligned}
\Pi^{ab}_{\mu\nu,\text{ loops}}(k)&=\frac{g^2}{8\pi^2}\delta^{ab}\left(\frac{5}{3}T(A)-\frac{4}{3}n_FT(R)\right)(k^2g_{\mu\nu}-k_\mu k_\nu)\left(\frac{1}{\epsilon}+\cdots\right)\\
&=\delta^{ab}(k^2g_{\mu\nu}-k_\mu k_\nu)\Pi_{\text{loops}}(k^2)
\end{aligned} \tag{4.39}
$$

and finally,

$$
\begin{aligned}
\Pi(k^2)&=\Pi_{\text{loops}}(k^2)+\Pi_{\text{ct}}(k^2)\\
&=\left(\frac{5}{3}T(A)-\frac{4}{3}n_FT(R)\right)\frac{g^2}{8\pi^2}\frac{1}{\epsilon}-(Z_3-1).
\end{aligned} \tag{4.40}
$$

For $\Pi(k^2)$ to be finite, we must have

$$
Z_3=1+\left(\frac{5}{3}T(A)-\frac{4}{3}n_FT(R)\right)\frac{g^2}{8\pi^2}\frac{1}{\epsilon}+O(g^4). \tag{4.41}
$$

4.1.2 Corrections to the fermion propagator

At one loop, the fermion receives corrections from the following diagrams.

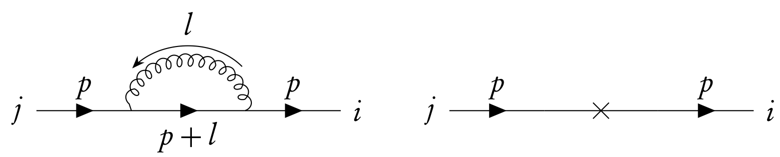


Figure 4.2: One loop corrections to the fermion propagator in spinor gauge theory.

For the first diagram, we have

$$i\Sigma_{\text{loop}}(\not{p}) = (ig)^2\left(\frac{1}{i}\right)^2 (T^a_R T^a_R)_{ij} \int \frac{d^4 l}{(2\pi)^4} \frac{\gamma^\nu(-\not{l}-\not{p}+m)\gamma_\nu}{l^2((l+p)^2+m^2)}. \tag{4.42}$$

Other than the colour factor $(T^a T^a)_{ij} = C(R)\delta_{ij}$, the expression is identical to the similar electrodynamics diagram,

$$i\Sigma_{\text{loop}}(\not{p}) = -C(R)\delta_{ij}(\not{p}+4m)\frac{ig^2}{8\pi^2}\frac{1}{\epsilon}. \tag{4.43}$$

With the counterterm, we have

$$\Sigma(\not{p}) = -(Z_2-1)\not{p}\delta_{ij} - (Z_m-1)m\delta_{ij} - C(R)\delta_{ij}(\not{p}+4m)\frac{g^2}{8\pi^2}\frac{1}{\epsilon}. \tag{4.44}$$

Absorbing divergences in the Z factors, we have

$$Z_2 = 1 - C(R)\frac{g^2}{8\pi^2}\frac{1}{\epsilon} + O(g^4) \quad \text{and} \quad Z_m = 1 - C(R)\frac{g^2}{2\pi^2}\frac{1}{\epsilon} + O(g^4). \tag{4.45}$$

Each family of fermions will receive an identical contribution to the divergent part of the Z factors.

4.1.3 Corrections to the vertex

At one loop, the fermion-fermion-gluon vertex receives corrections from the following diagrams.

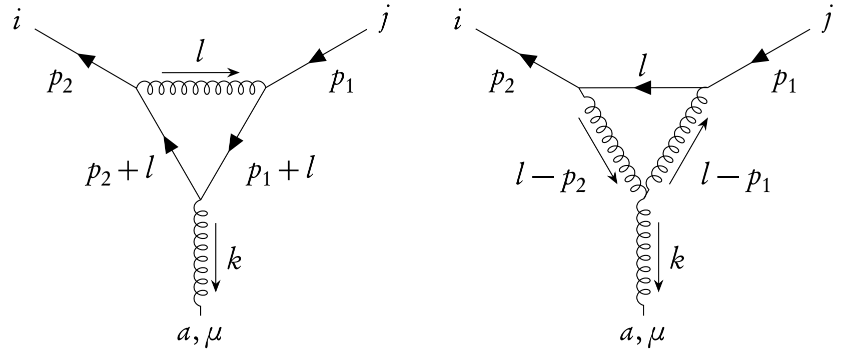


Figure 4.3: One loop contribution to the spinor-spinor-gluon vertex in spinor gauge theory.

For the first diagram, we have

$$i\mathbf{V}^{a\mu}_{ij,1} = (ig)^3\left(\frac{1}{i}\right)^3 (T^b_R T^a_R T^b_R)_{ij} \int \frac{d^4 l}{(2\pi)^4} \frac{\gamma^\nu(-\not{l}+m)\gamma^\mu(-\not{l}+m)\gamma_\nu}{l^2(l^2+m^2)(l^2+m^2)} \tag{4.46}$$

As the divergent part of the diagram is independent of the external momenta, we have set $p_1 = p_2 = k = 0$. Apart from the colour factor,

$$
\begin{aligned}
T_R^b T_R^a T_R^b &= ([T_R^b, T_R^a] + T_R^a T_R^b) T_R^b \\
&= i f^{bac} T_R^c T_R^b + T_R^a (T_R^b T_R^b) \\
&= \frac{1}{2} i f^{bac} [T_R^c, T_R^b] + C(R) T_R^a \\
&= -\frac{1}{2} f^{bac} f^{cbd} T_R^d + C(R) T_R^a \\
&= -\frac{1}{2} (-i f^{abc})(-i f^{dcb}) T_R^d + C(R) T_R^a \\
&= -\frac{1}{2} \operatorname{Tr}\Big(T_A^a T_A^d \Big) T_R^d + C(R) T_R^a \\
&= -\frac{1}{2} T(A) \delta^{ad} T_R^d + C(R) T_R^a \\
&= \Big(C(R) - \frac{1}{2} T(A) \Big) T_R^a,
\end{aligned} \tag{4.47}
$$

the diagram is identical to the similar diagram in electrodynamics, and we have

$$
\begin{aligned}
i \mathbf{V}^{a\mu}_{ij,\,1} &= \Big(C(R) - \frac{1}{2} T(A) \Big) \gamma^\mu (T_R^a)_{ij} \frac{i g^3}{8\pi^2} \frac{1}{\epsilon} \\
&= i g \gamma^\mu (T_R^a)_{ij} \Big(C(R) - \frac{1}{2} T(A) \Big) \frac{g^2}{8\pi^2} \frac{1}{\epsilon}.
\end{aligned} \tag{4.48}
$$

For the second diagram, we have

$$
i \mathbf{V}^{a\mu}_{ij,\,2} = (i g)^2 \Big(\frac{1}{i} \Big)^3 (T_R^b T_R^c)_{ij} \int \frac{d^4 l}{(2\pi)^4} \frac{i \mathbf{V}^{abc\mu\nu\rho}(k, -l, l) \gamma_\nu (-\not{l} + m) \gamma_\rho}{l^2 l^2 (l^2 + m^2)}, \tag{4.49}
$$

where

$$
i \mathbf{V}^{abc\mu\nu\rho}(p, q, r) = g f^{abc} \Big[(q - r)^\mu g^{\nu\rho} + (r - p)^\nu g^{\rho\mu} + (p - q)^\rho g_{\mu\nu} \Big]. \tag{4.50}
$$

Again, as we are only interested in the diverging part of the integral, we have set $p_1 = p_2 = k = 0$. Expanding the numerator, we have

$$
\begin{aligned}
&(T_R^b T_R^c)_{ij} i \mathbf{V}^{abc\mu\nu\rho}(k, -l, l) \gamma_\nu (-\not{l} + m) \gamma_\rho \\
&\qquad = g (T_R^b T_R^c)_{ij} f^{abc} [-2 l^\mu g^{\nu\rho} + (l - k)^\nu g^{\rho\mu} + (l + k)^\rho g^{\mu\nu}] \gamma_\nu (-\not{l} + m) \gamma_\rho \\
&\qquad = g (T_R^b T_R^c)_{ij} f^{abc} N^\mu.
\end{aligned} \tag{4.51}
$$

The colour factor is simplifies as

$$\begin{aligned} f^{abc}T_R^bT_R^c &= \frac{1}{2}f^{abc}[T_R^b, T_R^c] \\ &= \frac{1}{2}if^{abc}f^{bce}T_R^e \\ &= \frac{1}{2}i(-if^{abc})(-if^{ecb})T_R^e \\ &= \frac{1}{2}i\,\mathrm{Tr}\big(T_A^aT_A^e\big)T_R^e \\ &= \frac{1}{2}iT(A)T_R^a. \end{aligned} \tag{4.52}$$

To simplify N^μ we only keep terms quadratic in l, because we are interested only in the diverging part of the integral,

$$\begin{aligned} N^\mu &= [-2l^\mu g^{\nu\rho} + (l-k)^\nu g^{\rho\mu} + (l+k)^\rho g^{\mu\nu}]\gamma_\nu(-\not{l} + m)\gamma_\rho \\ &= 2l^\mu\gamma_\nu\not{l}\gamma^\nu - \not{l}\not{l}\gamma^\mu - \not{l}\not{l}\gamma^\mu \\ &= 2(d-2)l^\mu\not{l} + 2l^2\gamma^\mu \\ &\to \frac{2(d-2)}{d}l^2\gamma^\mu + 2l^2\gamma^\mu, \end{aligned} \tag{4.53}$$

where we used $\gamma_\nu\not{a}\gamma^\nu = (d-2)\not{a}$ and $l_\mu l_\nu \to d^{-1}l^2g_{\mu\nu}$. Since we are interested only in the divergent part of the integral, we put $d = 4$ and

$$N^\mu \to 3l^2\gamma^\mu. \tag{4.54}$$

Putting everything together, we have

$$\begin{aligned} i\mathbf{V}^{a\mu}_{ij,2} &= \frac{3}{2}(ig)^3\left(\frac{1}{i}\right)^3 T(A)(T_R^a)_{ij}\gamma^\mu\int\frac{d^4l}{(2\pi)^4}\frac{l^2}{l^2l^2(l^2+m^2)} \\ &= \frac{3}{2}g^3T(A)(T_R^a)_{ij}\gamma^\mu\left(\frac{i}{8\pi^2}\frac{1}{\epsilon} + \cdots\right) \\ &= ig\gamma^\mu(T_R^a)_{ij}T(A)\frac{3g^2}{16\pi^2}\left(\frac{1}{\epsilon} + \cdots\right). \end{aligned} \tag{4.55}$$

Finally,

$$\begin{aligned} i\mathbf{V}^{a\mu}_{ij} &= iZ_1g\gamma^\mu(T_R^a)_{ij} + i\mathbf{V}^{a\mu}_{ij,1} + i\mathbf{V}^{a\mu}_{ij,2} \\ &= ig\gamma^\mu(T_R^a)_{ij}\left(Z_1 + [C(R) + T(A)]\frac{g^2}{8\pi^2}\frac{1}{\epsilon}\right), \end{aligned} \tag{4.56}$$

and absorbing the divergence in Z_1 gives

$$Z_1 = 1 - [C(R) + T(A)]\frac{g^2}{8\pi^2}\frac{1}{\epsilon} + O(g^4). \tag{4.57}$$

4.1.4 Beta function

Relations between bare and renormalized parameters and fields for nonabelian gauge theory is the same as in electrodynamics. In particular,

$$m_0 = Z_2^{-1} Z_m m, \tag{4.58}$$

$$g_0 = Z_3^{-1/2} Z_2^{-1} Z_1 g. \tag{4.59}$$

If we define $\alpha = g^2/4\pi$, then

$$\alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^2 \alpha. \tag{4.60}$$

Let $G(\alpha,\epsilon) = \ln\left(Z_3^{-1} Z_2^{-2} Z_1^2\right)$, then G can be expressed as a power series in $1/\epsilon$,

$$G(\alpha,\epsilon) = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\epsilon^n}. \tag{4.61}$$

By the analysis of the previous section,

$$\frac{da}{d\ln\mu} = -\epsilon\alpha + \beta(\alpha), \tag{4.62}$$

where

$$\beta(\alpha) = \alpha^2 G_1'(\alpha). \tag{4.63}$$

We have calculated the Z factors to one-loop,

$$Z_1 = 1 - [C(R) + T(A)]\frac{\alpha}{2\pi}\frac{1}{\epsilon} + O(\alpha^2) \tag{4.64}$$

$$Z_2 = 1 - C(R)\frac{\alpha}{2\pi}\frac{1}{\epsilon} + O(\alpha^2) \tag{4.65}$$

$$Z_m = 1 - C(R)\frac{2\alpha}{\pi}\frac{1}{\epsilon} + O(\alpha^2) \tag{4.66}$$

$$Z_3 = 1 + \left(\frac{5}{3}T(A) - \frac{4}{3}n_F T(R)\right)\frac{\alpha}{2\pi}\frac{1}{\epsilon} + O(\alpha^2). \tag{4.67}$$

Using the values of Z factors above,

$$\begin{aligned}
G(\alpha,\epsilon) &= \ln Z_3^{-1} Z_2^{-2} Z_1^2 \\
&= \ln\left(1 - \left[\frac{5}{3}T(A) - \frac{4}{3}n_F T(R)\right]\frac{\alpha}{2\pi\epsilon}\right)\left(1 + C(R)\frac{\alpha}{\pi\epsilon}\right)\left(1 - [C(R) + T(A)]\frac{\alpha}{\pi\epsilon}\right) \\
&= \left(-\left[\frac{5}{3}T(A) - \frac{4}{3}n_F T(R)\right]\frac{\alpha}{2\pi} + C(R)\frac{\alpha}{\pi} - C(R)\frac{\alpha}{\pi} - T(A)\frac{\alpha}{\pi}\right)\frac{1}{\epsilon} + \cdots \\
&= \left[-\frac{11}{3}T(A) + \frac{4}{3}n_F T(R)\right]\frac{\alpha}{2\pi}\frac{1}{\epsilon} + \cdots,
\end{aligned} \tag{4.68}$$

so that

$$G_1(\alpha) = \left[-\frac{11}{3}T(A) + \frac{4}{3}n_F T(R)\right]\frac{\alpha}{2\pi} + O(\alpha^2), \tag{4.69}$$

and the beta function,

$$\beta(\alpha) = -\left[\frac{11}{3}T(A) - \frac{4}{3}n_F T(R)\right]\frac{\alpha^2}{2\pi} + O(\alpha^3) \tag{4.70}$$

or equivalently,

$$\beta(g) = -\left[\frac{11}{3}T(A) - \frac{4}{3}n_F T(R)\right]\frac{g^3}{16\pi^2} + O(g^5). \tag{4.71}$$

4.1.5 Anomalous dimension of mass

Anomalous dimension of mass is defined as

$$\gamma_m(\alpha) = \frac{d\ln m}{d\ln \mu}. \tag{4.72}$$

By the analysis of previous section, if $A(\alpha,\epsilon) = \ln Z_2^{-1} Z_m$, then

$$A(\alpha,\epsilon) = \sum_{n=1}^{\infty}\frac{A_n(\alpha)}{\epsilon^n}, \tag{4.73}$$

and the anomalous dimension is given by,

$$\gamma_m(\alpha) = \alpha A_1'(\alpha). \tag{4.74}$$

With Z factors as above, we have

$$\begin{aligned} A(\alpha,\epsilon) &= \ln Z_2^{-1} Z_m \\ &= \ln\left(1 + C(R)\frac{\alpha}{2\pi}\frac{1}{\epsilon}\right)\left(1 - C(R)\frac{2\alpha}{\pi}\frac{1}{\epsilon}\right) \\ &= -C(R)\frac{3\alpha}{2\pi}\frac{1}{\epsilon} + \cdots, \end{aligned} \tag{4.75}$$

so that

$$A_1(\alpha) = -C(R)\frac{3\alpha}{2\pi} + O(\alpha^2), \tag{4.76}$$

and

$$\gamma_m(\alpha) = -C(R)\frac{3\alpha}{2\pi} + O(\alpha^2) \tag{4.77}$$

4.1.6 Anomalous dimension of fields

Anomalous dimension of the field is defined as

$$\begin{aligned}\gamma_\psi(\alpha) &= \frac{1}{2}\frac{d\ln Z_2}{d\ln\mu}\\ &= \frac{1}{2}\frac{\partial \ln Z_2}{\partial\alpha}\frac{d\alpha}{d\ln\mu}\\ &= \frac{1}{2}\frac{\partial}{\partial\alpha}\Big(-C(R)\frac{\alpha}{2\pi}\frac{1}{\epsilon}\Big)(-\epsilon\alpha+\beta(\alpha))\\ &= C(R)\frac{\alpha}{4\pi}+O(\alpha^2).\end{aligned} \tag{4.78}$$

Similarly for the gauge field, we have

$$\begin{aligned}\gamma_A(\alpha) &= \frac{1}{2}\frac{d\ln Z_3}{d\ln\mu}\\ &= \frac{1}{2}\frac{\partial \ln Z_3}{\partial\alpha}\frac{d\alpha}{d\ln\mu}\\ &= \frac{1}{2}\frac{\partial}{\partial\alpha}\Big(\Big[\frac{5}{3}T(A)-\frac{4}{3}n_F T(R)\Big]\frac{\alpha}{2\pi}\frac{1}{\epsilon}\Big)(-\epsilon\alpha+\beta(\alpha))\\ &= -\Big[\frac{5}{3}T(A)-\frac{4}{3}n_F T(R)\Big]\frac{\alpha}{4\pi}+O(\alpha^2).\end{aligned} \tag{4.79}$$

4.2 Coupled to scalars

As before, we start with a manifestly gauge covariant Lagrangian for complex scalar fields in a representation R of the gauge group,

$$L_{\text{scalar}} = -(D^\mu\phi)^{\dagger j}(D_\mu\phi)_j - M^2\phi^{\dagger j}\phi_j - \frac{1}{4}\lambda\phi^{\dagger i}\phi_i\phi^{\dagger j}\phi_j. \tag{4.80}$$

We have $(D^\mu\phi)^{\dagger j} = \partial^\mu\phi^{\dagger j} + ig\phi^{\dagger k}A^a_\mu(T^a_R)_k{}^j$, and therefore

$$\begin{aligned}&(D^\mu\phi)^{\dagger j}D_\mu\phi_j\\ &\quad= -igA^a_\mu\Big[(\partial^\mu\phi^{\dagger j})(T^a_R)_j^k\phi_k - \phi^{\dagger j}(T^a_R)_j^k(\partial^\mu\phi_k)\Big] + g^2A^{a\mu}A^b_\mu\phi^{\dagger k}(T^a_R T^b_R)_k^l\phi_l.\end{aligned}$$

After expanding the kinetic term for gauge fields, adding a gauge fixing term, ghosts, inserting appropriate Z factors and organizing the Lagrangian as before, we have

$$\begin{aligned} L_0 =& +\frac{1}{2}A^{a\mu}(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)A^{a\nu}+\frac{1}{2}\xi^{-1}A^{a\mu}\partial_\mu\partial_\nu A^{a\nu}-\partial^\mu\bar{c}^a\partial_\mu c^a \\ & -\partial^\mu\phi^{\dagger j}\partial_\mu\phi_j-M^2\phi^{\dagger j}\phi_j \end{aligned} \tag{4.81}$$

$$\begin{aligned} L_1 =& -Z_{3g}\,g f^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^{c\nu}-\frac{1}{4}Z_{4g}\,g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu \\ & +Z_{1'}\,g f^{abc}A^a_\mu\partial^\mu\bar{c}^b c^c+igZ_1A^a_\mu\Big[(\partial^\mu\phi^{\dagger j})(T^a_R)_j^{\;k}\phi_k-\phi^{\dagger j}(T^a_R)_j^{\;k}(\partial^\mu\phi_k)\Big] \\ & -Z_{4g}\,g^2A^{a\mu}A^b_\mu\phi^{\dagger j}(T^a_RT^b_R)_j^{\;k}\phi_k-\frac{1}{4}Z_\lambda\lambda\phi^{\dagger j}\phi_j\phi^{\dagger k}\phi_k+L_{\text{ct}} \end{aligned} \tag{4.82}$$

$$\begin{aligned} L_{\text{ct}} =& +\frac{1}{2}(Z_3-1)A^{a\mu}(g_{\mu\nu}-\partial_\mu\partial_\nu)A^{a\nu}-(Z_{2'}-1)\partial^\mu\bar{c}^a\partial_\mu c^a \\ & -(Z_2-1)\partial^\mu\phi^{\dagger j}\partial_\mu\phi_j-(Z_M-1)M^2\phi^{\dagger j}\phi_j, \end{aligned} \tag{4.83}$$

so that the complete Lagrangian, $L=L_0+L_1$.

4.2.1 Corrections to the gluon propagator

We have the following diagrams correcting the gluon propagator in scalar nonabelian gauge theory.

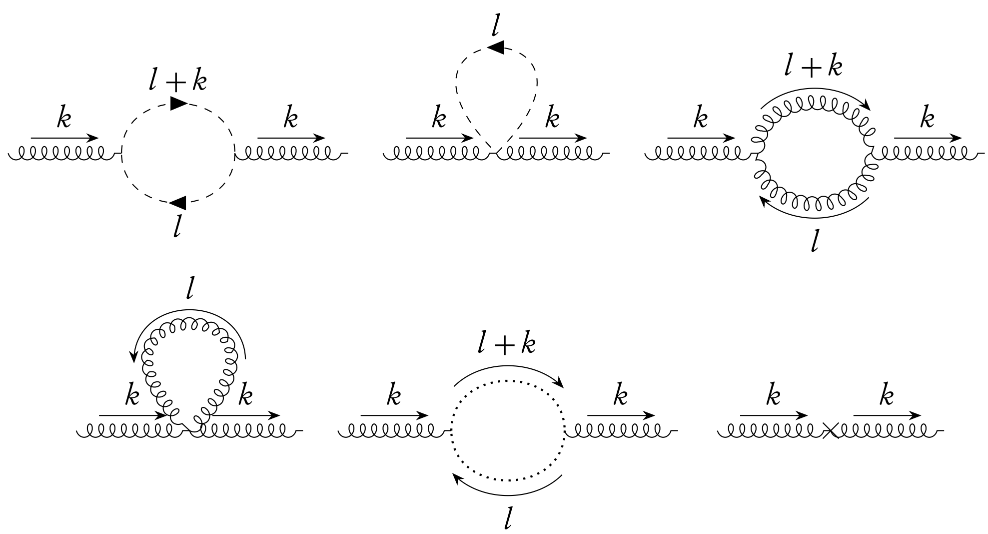


Figure 4.4: One loop corrections to the gluon propagator in scalar gauge theory.

The third, fourth and fifth diagrams are evaluated exactly as with spinor fields. Third diagram gives

$$i\Pi^{ab}_{\mu\nu,\,\text{gluon loop}}(k)=\frac{ig^2}{16\pi^2}T(A)\delta^{ab}\Big(\frac{19}{6}k^2g_{\mu\nu}-\frac{11}{3}k_\mu k_\nu\Big)\Big(\frac{1}{\epsilon}+\cdots\Big), \tag{4.84}$$

the fourth diagram vanishes, and the fifth diagram gives

$$i\Pi^{ab}_{\mu\nu,\,\text{ghost loop}}(k)=\frac{ig^2}{8\pi^2}T(A)\delta^{ab}\Big(\frac{1}{12}k^2g_{\mu\nu}\frac{1}{6}k_\mu k_\nu\Big)\Big(\frac{1}{\epsilon}+\cdots\Big). \tag{4.85}$$

Adding up the two contributions,

$$i\Pi^{ab}_{\mu\nu,\text{ gluon loop}} + i\Pi^{ab}_{\mu\nu,\text{ ghost loop}} = T(A)\frac{5ig^2}{24\pi^2}\frac{1}{\epsilon}\delta^{ab}\left(k^2 g_{\mu\nu} - k_\mu k_\nu\right). \tag{4.86}$$

For the first two diagrams, we have

$$\begin{aligned} i\Pi^{ab}_{\mu\nu,\text{ scalar loop}} =&(ig)^2\,\mathrm{Tr}\left(T^a_R T^b_R\right)\left(\frac{1}{i}\right)^2\int\frac{d^4l}{(2\pi)^4}\frac{(2l+k)^\mu(2l+k)^\nu}{(l^2+M^2)((l+k)^2+M^2)} \\ &+(-ig^2)2\,\mathrm{Tr}\left(T^a_R T^b_R\right)\left(\frac{1}{i}\right)\int\frac{d^4l}{(2\pi)^4}\frac{1}{l^2+M^2}. \end{aligned} \tag{4.87}$$

Other than a color factor of $\mathrm{Tr}\left(T^a_R T^b_R\right) = T(R)\delta^{ab}$, the integrals are identical to similar ones in scalar electrodynamics. Therefore

$$i\Pi^{ab}_{\mu\nu,\text{ scalar loop}} = -T(R)\frac{ig^2}{24\pi^2}\frac{1}{\epsilon}\delta^{ab}(k^2 g^{\mu\nu} - k^\mu k^\nu). \tag{4.88}$$

We notice that $\Pi^{ab}_{\mu\nu} = \Pi(k^2)\delta^{ab}(k^2 g_{\mu\nu} - k_\mu k_\nu)$. Putting everything together, we have

$$\begin{aligned} \Pi(k^2)_{1\text{ loop}} &= \Pi(k^2)_{\text{loop}} + \Pi(k^2)_{\text{ct}} \\ &= \left[\frac{5}{3}T(A) - \frac{1}{3}T(R)\right]\frac{g^2}{8\pi^2}\frac{1}{\epsilon} - (Z_3 - 1) + O(\epsilon^0). \end{aligned} \tag{4.89}$$

To cancel the divergence

$$Z_3 = 1 + \left[\frac{5}{3}T(A) - \frac{1}{3}T(R)\right]\frac{g^2}{8\pi^2}\frac{1}{\epsilon}. \tag{4.90}$$

4.2.2 Corrections to the scalar propagator

At one loop, the following diagrams contribute to the scalar propagator.

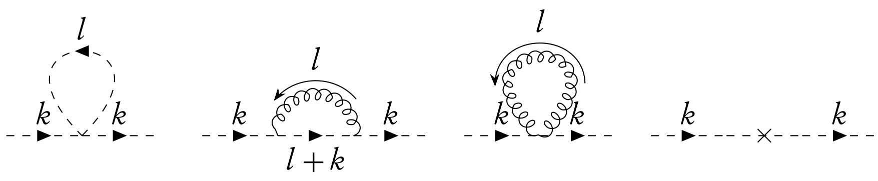


Figure 4.5: One loop corrections to the scalar propagator in scalar gauge theory.

The first diagram is exactly the same as for scalar electrodynamics (and pure phi^4-theory). Integrals for the second and third diagrams are

$$
\begin{aligned}
i\Pi(k^2)_i^{\,j} = {} & \delta_i^{\,j}(\cdots) \\
& + (T_R^a T_R^a)_i^{\,j}(ig)^2\left(\frac{1}{i}\right)^2 \int \frac{d^4 l}{(2\pi)^4}\frac{(l+2k)^\mu P_{\mu\nu}(l)(l+2k)^\nu}{l^2((l+k)^2+M^2)} && (4.91)\\
& + 2(T_R^a T_R^a)_i^{\,j}(-ig^2)\left(\frac{1}{i}\right)\int \frac{d^4 l}{(2\pi)^4}\frac{g_{\mu\nu}P^{\mu\nu}(l)}{l^2+m_g^2} && (4.92)\\
& - i(Z_2-1)k^2 - i(Z_M-1)M^2 && (4.93)
\end{aligned}
$$

Other than a color factor, the integrals are same as the ones encountered in scalar electrodynamics. The third integral vanishes in the $m_g \to 0$ limit. Simplifying the color factor $(T_R^a T_R^a)_i^{\,j} = C(R)\delta_i^{\,j}$, and using previous results, we have

$$
\Pi(k^2)_i^{\,j} = \left(C(R)\frac{3g^2k^2}{8\pi^2} + \frac{\lambda M^2}{8\pi^2}\right)\frac{1}{\epsilon} - (Z_2-1)k^2 - (Z_M-1)M^2, \tag{4.94}
$$

so that to keep $\Pi(k^2)$ finite, we must have

$$
Z_2 = 1 + C(R)\frac{3g^2}{8\pi^2}\frac{1}{\epsilon} \quad \text{and} \quad Z_M = 1 + \frac{\lambda}{8\pi^2}\frac{1}{\epsilon}. \tag{4.95}
$$

4.2.3 Corrections to the scalar-scalar-gluon vertex

At one loop, we have the following corrections.

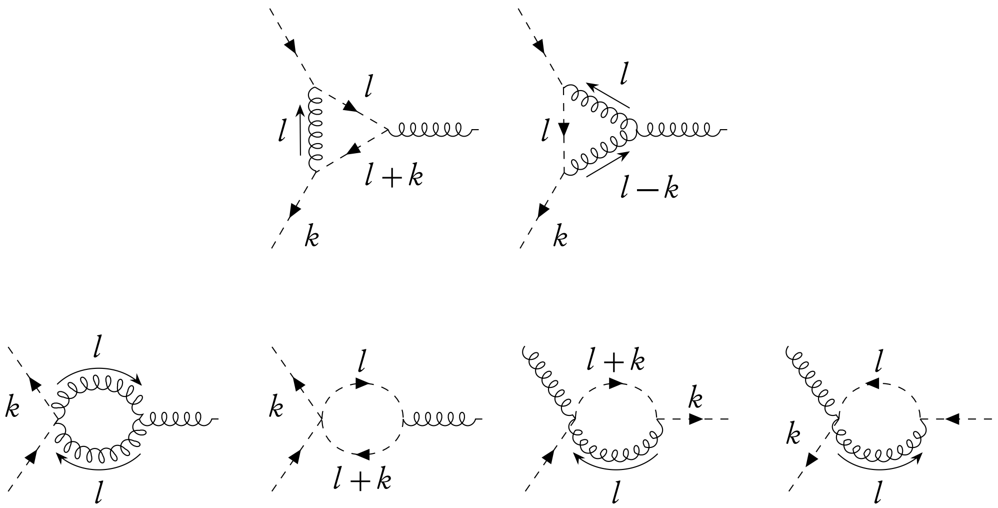


Figure 4.6: One loop corrections to the scalar-scalar-gluon vertex in scalar gauge theory.

The first, fourth, fifth and sixth diagrams will be proportional to their counterparts in scalar electrodynamics. Hence, the first, fourth and sixth diagrams vanish. In the second diagram there is a vertex in which an external scalar with zero momentum is connected to an internal gluon with momentum l. This would lead to a factor of $igl^\nu P_{\nu\rho}(l)$, and the corresponding integral vanishes.

For the third diagram, the three gluon vertex will give a factor of f^{abc} and the gluon scalar-scalar-gluon-gluon vertex will give the color factor $(T_R^a T_R^b + T_R^a T_R^b)$. We note that f^{abc} is antisymmetric in indices a, b while $T_R^a T_R^b + T_R^a T_R^b$ is symmetric, hence this diagram will vanish as well.

Finally, consider the fifth diagram. Applying Feynman rules to this diagram gives

$$(-ig^2)(ig)T_R^b(T_R^a T_R^b + T_R^b T_R^a)\left(\frac{1}{i}\right)\int \frac{d^4 l}{(2\pi)^4}\frac{P^{\mu\nu}(l)(l+2k)_\nu}{l^2((l+k)^2+M^2)}. \tag{4.96}$$

Other than the color factor,

$$\begin{aligned} T_R^b(T_R^a T_R^b + T_R^b T_R^a) &= T_R^b T_R^a T_R^b + T_R^b T_R^b T_R^a \\ &= \left[C(R) - \frac{1}{2}T(A)\right]T_R^a + C(R)T_R^a \\ &= 2\left[C(R) - \frac{1}{4}T(A)\right]T_R^a, \end{aligned} \tag{4.97}$$

the integral is otherwise identical to the one encountered in scalar electrodynamics. Therefore, we have

$$i\mathbf{V}_3^{a\mu} = iZ_1 g T_R^a k^\mu - \left[C(R) - \frac{1}{4}T(A)\right]T_R^a \frac{3ig^3 k^\mu}{8\pi^2}\frac{1}{\epsilon}, \tag{4.98}$$

and therefore

$$Z_1 = 1 + \left[C(R) - \frac{1}{4}T(A)\right]\frac{3g^2}{8\pi^2}\frac{1}{\epsilon}. \tag{4.99}$$

4.2.4 Beta function

For reference, the Z factors for snonabelian gauge theory are

$$Z_1 = 1 + \left[C(R) - \frac{1}{4}T(A)\right]\frac{3g^2}{8\pi^2}\frac{1}{\epsilon} \tag{4.100}$$

$$Z_2 = 1 + C(R)\frac{3g^2}{8\pi^2}\frac{1}{\epsilon} \tag{4.101}$$

$$Z_3 = 1 + \left[\frac{5}{3}T(A) - \frac{1}{3}T(R)\right]\frac{g^2}{8\pi^2}\frac{1}{\epsilon} \tag{4.102}$$

$$Z_M = 1 + \frac{\lambda}{8\pi^2}\frac{1}{\epsilon}. \tag{4.103}$$

$$\tag{4.104}$$

As in scalar electrodynamics, the beta function is given by

$$\beta(g) = g\left(\frac{g}{2}\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial \lambda}\right)G_1(g,\lambda), \tag{4.105}$$

where $G(g,\lambda) = \ln\left(Z_3^{-1/2}Z_2^{-1}Z_1\right)$, and G_1 is the coefficient of $1/\epsilon$ when it is written as a power series in ϵ^{-1}.

Computing G we have

$$\ln\left(Z_3^{-1/2} Z_2^{-1} Z_1\right) = -\frac{1}{2}\ln Z_3 - \ln Z_2 + \ln Z_1$$
$$= \left(-\frac{5}{6}T(A) + \frac{1}{6}T(R) - 3C(R) + 3C(R) - \frac{3}{4}T(A)\right)\frac{g^2}{8\pi^2}\frac{1}{\epsilon}$$
$$= -\left(\frac{19}{3}T(A) - \frac{1}{3}T(R)\right)\frac{g^2}{16\pi^2}\frac{1}{\epsilon} + O(\epsilon^{-2}), \tag{4.106}$$

and therefore the beta function

$$\beta_g(g) = -\left[\frac{19}{3}T(A) - \frac{1}{3}T(R)\right]\frac{g^2}{16\pi^2}. \tag{4.107}$$

4.2.5 Anomalous dimensions

For anomalous dimension of the gauge field, we proceed as before

$$\gamma_A(g,\lambda) = \frac{1}{2}\frac{d\ln Z_3}{d\ln\mu} = [T(R) - 5T(A)]\frac{g^2}{48\pi^2}. \tag{4.108}$$

And similarly, the anomalous dimension of scalar field

$$\gamma_\phi(g,\lambda) = \frac{1}{2}\frac{d\ln Z_2}{d\ln\mu} = -C(R)\frac{3g^2}{16\pi^2}. \tag{4.109}$$

Finally, for the anomalous dimension of mass, we need

$$B(g,\lambda) = \ln\left(Z_M^{1/2} Z_2^{-1/2}\right) = \left(\frac{\lambda}{16\pi^2} - C(R)\frac{3g^2}{16\pi^2}\right)\frac{1}{\epsilon} + O(\epsilon^{-2}) \tag{4.110}$$

and

$$\gamma_M(g,\lambda) = \left(\frac{g}{2}\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial\lambda}\right)B_1(g,\lambda) = \frac{\lambda}{16\pi^2} - C(R)\frac{3g^2}{16\pi^2}. \tag{4.111}$$

Chapter 5

Summary

All the results derived in preceeding chapters are repeated here once again for reference.

Scalar field renormalization

ϕ^3 theory in $d=6$,	$Z_\phi = 1 - \frac{\varkappa^2}{384\pi^3}\frac{1}{\epsilon}$
ϕ^4 theory in $d=4$,	$Z_\phi = 1 + O(\lambda^2)$
Yukawa theory,	$Z_\phi = 1 - \frac{g^2}{4\pi^2}\frac{1}{\epsilon}$
Scalar electrodynamics,	$Z_2 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}$
Scalar gauge theory,	$Z_2 = 1 + C(R)\frac{3g^2}{8\pi^2}\frac{1}{\epsilon}$

Anomalous dimension of scalar field

ϕ^3 theory in $d=6$,	$\gamma_\phi = \frac{\varkappa^2}{384\pi^3}$
ϕ^4 theory in $d=4$,	$\gamma_\phi = O(\lambda^2)$
Yukawa theory,	$\gamma_\phi = \frac{g^2}{8\pi^2}$
Scalar electrodynamics,	$\gamma_\phi = -\frac{3e^2}{8\pi^2}$
Scalar gauge theory,	$\gamma_\phi = -C(R)\frac{3e^2}{8\pi^2}$

Scalar mass renormalization

ϕ^3 theory in $d=6$, $\quad Z_M = 1 - \frac{\varkappa^2}{64\pi^3}\frac{1}{\epsilon}$

ϕ^4 theory in $d=4$, $\quad Z_M = 1 + \frac{\lambda}{16\pi^2}\frac{1}{\epsilon}$

Yukawa theory, $\quad Z_M = 1 + \left(\frac{\lambda}{16\pi^2} + \frac{\varkappa^2}{16\pi^2 M^2} - \frac{3g^2m^2}{2\pi^2 M^2}\right)\frac{1}{\epsilon}$

Scalar electrodynamics, $\quad Z_M = 1 + \frac{\lambda}{8\pi^2}\frac{1}{\epsilon}$

Scalar gauge theory, $\quad Z_M = 1 + \frac{\lambda}{8\pi^2}\frac{1}{\epsilon}$

Anomalous dimension of scalar mass

ϕ^3 theory in $d=6$, $\quad \gamma_M = -\frac{5\varkappa^2}{768\pi^3}$

ϕ^4 theory in $d=4$, $\quad \gamma_M = \frac{\lambda}{32\pi^2}$

Yukawa theory, $\quad \gamma_M = \frac{\lambda}{32\pi^2} + \frac{\varkappa}{32\pi^2 M^2} + \left(1 - \frac{6m^2}{M^2}\right)\frac{g^2}{8\pi^2}$

Scalar electrodynamics, $\quad \gamma_M = \frac{\lambda}{16\pi^2} - \frac{3e^2}{8\pi^2}$

Scalar gauge theory, $\quad \gamma_M = \frac{\lambda}{16\pi^2} - C(R)\frac{3e^2}{8\pi^2}$

Spinor field renormalization

Yukawa theory, $\quad Z_\psi = 1 - \frac{g^2}{16\pi^2}\frac{1}{\epsilon}$

Spinor electrodynamics, $\quad Z_2 = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon}$

Spinor gauge theory, $\quad Z_2 = 1 - C(R)\frac{g^2}{8\pi^2}\frac{1}{\epsilon}$

Anomalous dimension of spinor field

Yukawa theory, $\quad \gamma_\psi = \frac{g^2}{32\pi^2}$

Spinor electrodynamics, $\quad \gamma_\psi = \frac{e^2}{8\pi^2}$

Spinor gauge theory, $\quad \gamma_\psi = C(R)\frac{g^2}{16\pi^2}$

Spinor mass renormalization

Yukawa theory,	$Z_m = 1 + \frac{g^2}{8\pi^2}\frac{1}{\epsilon}$
Spinor electrodynamics,	$Z_m = 1 - \frac{e^2}{2\pi^2}\frac{1}{\epsilon}$
Spinor gauge theory,	$Z_m = 1 - C(R)\frac{g^2}{2\pi^2}\frac{1}{\epsilon}$

Anomalous dimension of spinor mass

Yukawa theory,	$\gamma_m = +\frac{g^2}{16\pi^2}$
Spinor electrodynamics,	$\gamma_m = -\frac{3e^2}{8\pi^2}$
Spinor gauge theory,	$\gamma_m = -C(R)\frac{3g^2}{8\pi^2}$

Gauge field renormalization

Spinor electrodynamics,	$Z_3 = 1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon}$
Scalar electrodynamics,	$Z_3 = 1 - \frac{e^2}{24\pi^2}\frac{1}{\epsilon}$
Spinor gauge theory,	$Z_3 = 1 + \left(\frac{5}{3}T(A) - \frac{4}{3}T(R)\right)\frac{g^2}{8\pi^2}\frac{1}{\epsilon}$
Scalar gauge theory,	$Z_3 = 1 + \left(\frac{5}{3}T(A) - \frac{1}{3}T(R)\right)\frac{g^2}{8\pi^2}\frac{1}{\epsilon}$

Anomalous dimension of gauge field

Spinor electrodynamics,	$\gamma_A = \frac{e^2}{6\pi^2}$
Scalar electrodynamics,	$\gamma_A = \frac{e^2}{48\pi^2}$
Spinor gauge theory,	$\gamma_A = -\left(\frac{5}{3}T(A) - \frac{4}{3}T(R)\right)\frac{g^2}{16\pi^2}$
Scalar gauge theory,	$\gamma_A = -(5T(A) - T(R))\frac{g^2}{48\pi^2}$

Coupling renormalization

ϕ^3 theory in $d=6$,	$Z_\varkappa = 1 - \frac{\varkappa^2}{64\pi^3}\frac{1}{\epsilon}$
ϕ^4 theory in $d=4$,	$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2}\frac{1}{\epsilon}$
Yukawa interaction,	$Z_g = 1 + \frac{g^2}{8\pi^2}\frac{1}{\epsilon}$
ϕ^3 coupling in Yukawa theory,	$Z_\varkappa = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3mg^3}{\pi^2\varkappa}\right)\frac{1}{\epsilon}$
ϕ^4 coupling in Yukawa theory,	$Z_\lambda = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3g^4}{\pi^2\lambda}\right)\frac{1}{\epsilon}$
Spinor electrodynamics,	$Z_1 = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon}$
Scalar electrodynamics,	$Z_1 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}$
Spinor gauge theory,	$Z_1 = 1 - (C(R) + T(A))\frac{g^2}{8\pi^2}\frac{1}{\epsilon}$
Scalar gauge theory,	$Z_1 = 1 + \left(C(R) - \frac{1}{4}T(A)\right)\frac{3g^2}{8\pi^2}\frac{1}{\epsilon}$

Beta functions

ϕ^3 theory in $d=6$,	$\beta_\varkappa(\varkappa) = -\frac{3\varkappa^3}{256\pi^3}$
ϕ^4 theory in $d=4$,	$\beta_\lambda(\lambda) = \frac{3\lambda^2}{16\pi^2}$
Yukawa interaction,	$\beta_g(g,\varkappa,\lambda) = \frac{5g^3}{16\pi^2}$
ϕ^3 coupling in Yukawa theory,	$\beta_\varkappa(g,\varkappa,\lambda) = \frac{3g^2\varkappa}{8\pi^2} - \frac{3mg^3}{\pi^2} + \frac{3\varkappa\lambda}{16\pi^2}$
ϕ^4 coupling in Yukawa theory,	$\beta_\lambda(g,\varkappa,\lambda) = \frac{3\lambda^2}{16\pi^2} + \frac{g^2\lambda}{2\pi^2} - \frac{3g^4}{\pi^2}$
Spinor electrodynamics,	$\beta(e) = \frac{e^3}{12\pi^2}$
Scalar electrodynamics,	$\beta(e,\lambda) = \frac{e^3}{48\pi^2}$
Spinor gauge theory,	$\beta(g) = -\left(\frac{11}{3}T(A) - \frac{4}{3}T(R)\right)\frac{g^3}{16\pi^2}$
Scalar gauge theory,	$\beta(g,\lambda) = -\left(\frac{19}{3}T(A) - \frac{1}{3}T(R)\right)\frac{g^3}{16\pi^2}$

Part II
Amplitudes

Chapter 6
Spinor Helicity Formalism

After introducing spinor helicity variables as linearly independent solutions to the massless Dirac equation, we express things like momentum conservation, Fierz and Schouten identities, and polarization vectors in this new language. To demonstrate the power of this new formalism, Feynman rules for QED are written in terms of twistor variables and scattering amplitude for Compton scattering is computed using these new tools.

The free Dirac equation,

$$(-i\partial\!\!\!/ + m)\psi = 0, \tag{6.1}$$

has plane wave solutions of the form

$$\psi(x) = \sum_{s=\pm} \int \widetilde{dp} \Big[b_s(p) u_s(p) e^{ipx} + d_s^\dagger(p) v_s(p) e^{-ipx} \Big], \tag{6.2}$$

where $\widetilde{dp} = d^3p/(2\pi)^3 2E_p$ is the Lorentz invariant momentum measure and $b_\pm^\dagger(p)$, $d_\pm^\dagger(p)$ and $b_\pm(p)$, $d_\pm(p)$ are fermionic creation and annihilation operators respectively that take care of the Grassmann nature of $\psi(x)$. The four component spinors $u_\pm(p)$ and $v_\pm(p)$ are commuting and solve

$$(p\!\!\!/ + m)u_s(p) = 0 \quad \text{and} \quad (-p\!\!\!/ + m)v_s(p) = 0. \tag{6.3}$$

Due to the group theory relation

$$(2,1) \otimes (1,2) \otimes (2,2) = (1,1) \oplus (3,1) \oplus (1,3) \oplus (3,3), \tag{6.4}$$

for representations of the Lorentz group, there is an invariant symbol $\sigma^\mu_{a\dot{a}}$ which provides a dictionary between vector fields $A_\mu(x)$ and fields carrying one undotted and one dotted index $A_{a\dot{a}}(x)$,

$$A_{a\dot{a}}(x) = \sigma^\mu_{a\dot{a}} A_\mu(x). \tag{6.5}$$

A consistent choice for the invariant symbols is $\sigma^\mu = (I, \sigma^i)$ and $\bar{\sigma}^\mu = (I, -\sigma^i)$, where σ^i for $i = 1,2,3$ are Pauli matrices.

For a given four-momentum $p^\mu = (E, p^i)$ with $p^2 = -m^2$, we can define *momentum bispinors*

$$p_{a\dot{a}} = \sigma^\mu_{a\dot{a}} p_\mu \quad \text{and} \quad p^{\dot{a}a} = \bar{\sigma}^{\mu\dot{a}a} p_\mu, \tag{6.6}$$

which can be thought of 2×2 matrices

$$p_{a\dot{a}} = \begin{pmatrix} -p^0 + p^3 & p^1 - i p^2 \\ p^1 + i p^2 & -p^0 - p^3 \end{pmatrix} \quad \text{and} \quad p^{\dot{a}a} = \begin{pmatrix} -p^0 - p^3 & -p^1 + i p^2 \\ -p^1 - i p^2 & -p^0 + p^3 \end{pmatrix}. \quad (6.7)$$

The determinant is

$$\det p = -p^\mu p_\mu = m^2. \quad (6.8)$$

Moreover, with the convention for gamma-matrices, we have

$$\not{p} = \begin{pmatrix} 0 & p_{a\dot{a}} \\ p^{\dot{a}a} & 0 \end{pmatrix}. \quad (6.9)$$

6.1 Spinor helicity variables

In the extreme relativistic limit, when Mandelstam variables are much larger than fermion mass, we take $m \to 0$. For massless fermions, the four-component spinors satisfy

$$\not{p} v_s(p) = 0 \quad \text{and} \quad \bar{u}_s(p)\not{p} = 0, \quad (6.10)$$

and the index $s = \pm$ indicates the helicity $h = s/2$. Let

$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}, \quad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix} \quad (6.11)$$

and

$$\bar{u}_+(p) = \begin{pmatrix} [p|^a & 0 \end{pmatrix}, \quad \bar{u}_-(p) = \begin{pmatrix} 0 & \langle p|_{\dot{a}} \end{pmatrix} \quad (6.12)$$

solve the massless Dirac equation. The angle and square spinors are two-component commuting spinors that satisfy the massless Weyl equation,

$$p^{\dot{a}a}|p]_a = 0, \qquad p_{a\dot{a}}|p\rangle^{\dot{a}} = 0, \qquad [p|^a p_{a\dot{a}} = 0, \qquad \langle p|_{\dot{a}} p^{\dot{a}a} = 0. \quad (6.13)$$

These two-component commuting spinors are also sometimes called *twistors*. For real valued momenta, the Dirac equation has only two independent solutions and the angle and square spinors are related by

$$([p|^a)^* = |p\rangle^{\dot{a}} \quad \text{and} \quad (\langle p|_{\dot{a}})^* = |p]_a, \quad (6.14)$$

so that we have $v_s(p) = u_{-s}(p)$.

Using the spin-sum completeness relation for massless spinors, we have

$$-\not{p} = \sum_{s=\pm} u_s(p)\bar{u}_s(p) = |p\rangle[p| + |p]\langle p|. \quad (6.15)$$

On matching undotted and dotted indices with the matrix form of $\not{p}$ above,

$$p_{a\dot{a}} = -|p]_a \langle p|_{\dot{a}}, \quad p^{\dot{a}a} = -|p\rangle^{\dot{a}} [p|^a. \quad (6.16)$$

In explicit terms, for a (lightlike) momentum of the form

$$p^{\mu} = \begin{pmatrix} E & E\sin\theta\cos\phi & E\sin\theta\sin\phi & E\cos\theta \end{pmatrix}, \tag{6.17}$$

we have

$$\begin{aligned} p_{a\dot{a}} &= \begin{pmatrix} -p^0 + p^3 & p^1 - i p^2 \\ p^1 + i p^2 & -p^0 - p^3 \end{pmatrix} = E \begin{pmatrix} -1 + \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -1 - \cos\theta \end{pmatrix} \\ &= 2E \begin{pmatrix} -\sin^2\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & -\cos^2\frac{\theta}{2} \end{pmatrix}. \end{aligned} \tag{6.18}$$

Similarly,

$$p^{\dot{a}a} = -2E \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}. \tag{6.19}$$

Both $p_{a\dot{a}}$ and $p^{\dot{a}a}$ are matrices of rank 1 and as solutions of the massless Weyl equations, we have

$$|p] = \sqrt{2E} \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}, \quad |p\rangle = \sqrt{2E} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \tag{6.20}$$

and

$$\langle p| = \sqrt{2E} \begin{pmatrix} \sin\frac{\theta}{2} & -e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix}, \quad [p| = \sqrt{2E} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix}. \tag{6.21}$$

The factor of $\sqrt{2E}$ is arbitrary, but has been chosen so that the relations $p_{a\dot{a}} = |p]\langle p|$ and $p^{\dot{a}a} = |p\rangle [p|$ hold.

We can define spinor brackets by contracting indices appropriately,

$$\langle p\, q\rangle = \langle p|_{\dot{a}}\, |q\rangle^{\dot{a}}, \quad [p\, q] = [p|^{a} |q]_{a}. \tag{6.22}$$

There are no mixed brackets becaues spinor indices cannot be contracted to give a Lorentz scalar. Due to spinor indices, these brackets are antisymmetric

$$\langle p\, q\rangle = -\langle q\, p\rangle, \quad [p\, q] = -[q\, p]. \tag{6.23}$$

For real valued momenta we also have $[p\, q]^* = \langle q\, p\rangle$.

We also have the following relation

$$\begin{aligned} \langle p\, q\rangle [p\, q] &= -\langle p\, q\rangle [q\, p] \\ &= \langle p|_{\dot{a}}\, q^{\dot{a}b} |p]_b \\ &= -p_{b\dot{a}} q^{\dot{a}b} \\ &= -p_{\mu} q_{\nu} (\sigma^{\mu}_{b\dot{a}} \sigma^{\nu \dot{a}b}) \\ &= -p_{\mu} q_{\nu} (-2g^{\mu\nu}) \\ &= 2p\cdot q \end{aligned} \tag{6.24}$$

In terms of these new spinor helicity variables, momentum conservation can be written by noting (for a process with n external particles),

$$\sum_i \not{p}_i = -\sum_i (|i]\langle i| + |i\rangle [i|) = 0. \tag{6.25}$$

On matching spinor indices, we have

$$\sum_i |i]\langle i| = 0 \quad \text{and} \quad \sum_i |i\rangle [i| = 0. \tag{6.26}$$

Dotting with some spinors, we also have

$$\sum_i [k\, i]\langle i\, l\rangle = 0 \quad \text{and} \quad \sum_i \langle k\, i\rangle [i\, l] = 0. \tag{6.27}$$

In particular, for a four particle process, we have

$$|1]\langle 1| + |2]\langle 2| + |3]\langle 3| + |4]\langle 4| = 0. \tag{6.28}$$

Dotting with $[1|$ and $|2\rangle$, we get

$$[1\,3]\langle 3\,2\rangle + [1\,4]\langle 4\,2\rangle = 0, \tag{6.29}$$

and other similar identities. Using $(p+q)^2 = \langle p\,q\rangle[p\,q]$, we also have identities like,

$$\langle 1\,2\rangle[1\,2] = (p_1 + p_2)^2 = (p_3 + p_4)^2 = \langle 3\,4\rangle[3\,4]. \tag{6.30}$$

These identities will be used extensively to simplify expressions for amplitudes in terms of twistor brackets later.

6.1.1 Fierz and Schouten identities

We also have the Fierz identities, which are usually written in terms of four component spinors $u_s(p)$ and $v_s(p)$, in terms of twistor variables

$$\begin{aligned}\langle p|\gamma^\mu|q]\gamma_\mu &= \langle p|_{\dot a}\, \bar\sigma^{\mu\dot a b}|q]_b(\sigma_{\mu c\dot d} + \bar\sigma_\mu^{\dot c d}) \\ &= \langle p|_{\dot a}\, \bar\sigma^{\mu\dot a b}\sigma_{\mu c\dot d}|q]_b + \langle p|_{\dot a}\, \bar\sigma^{\mu\dot a b}\sigma_\mu^{\dot c d}|q]_b \\ &= -2\,\langle p|_{\dot a}\, \delta^{\dot a}_{\dot d}\delta^b_c|q]_b - 2\,\langle p|_{\dot a}\, \epsilon^{\dot a\dot c}\epsilon^{bd}|q]_b \\ &= -2|q]_c\,\langle p|_{\dot d} - 2\,|p\rangle^{\dot c}\,[q|^d,\end{aligned} \tag{6.31}$$

and therefore,

$$-\frac{1}{2}\,\langle p|\gamma^\mu|q]\gamma_\mu = |q]\,\langle p| + |p\rangle\,[q|. \tag{6.32}$$

Similarly,

$$-\frac{1}{2}[p|\gamma^\mu\,|q\rangle\,\gamma_\mu = |q\rangle\,[p| + |p]\,\langle q| \tag{6.33}$$

In an alternate form,

$$\langle p|\gamma^\mu|q]\langle r|\gamma_\mu|s] = 2\langle p\, r\rangle[q\, s], \tag{6.34}$$
$$[p|\gamma^\mu|q\rangle\langle r|\gamma_\mu|s] = 2[p\, s]\langle q\, r\rangle, \tag{6.35}$$
$$[p|\gamma^\mu|q\rangle[r|\gamma_\mu|s\rangle = 2[p\, r]\langle q\, s\rangle. \tag{6.36}$$

If we note that the twistors are elements of a complex two dimensional vector space, any three twistors $|i\rangle, |j\rangle, |k\rangle$ are going to be linarly dependent,

$$|k\rangle = a\,|i\rangle + b\,|j\rangle\,. \tag{6.37}$$

Dotting once by $\langle i|$ and once by $\langle j|$, we can solve for $a = \langle j\, k\rangle/\langle j\, i\rangle$ and $b = \langle i\, k\rangle/\langle i\, j\rangle$ so that

$$|i\rangle\langle j\, k\rangle + |j\rangle\langle k\, i\rangle + |k\rangle\langle i\, j\rangle = 0. \tag{6.38}$$

Dotting by an angle twistor $\langle l|$, we get the *Schouten identity*,

$$\langle l\, i\rangle\langle j\, k\rangle + \langle l\, j\rangle\langle k\, i\rangle + \langle l\, k\rangle\langle i\, j\rangle = 0. \tag{6.39}$$

Similarly for square twistors,

$$[l\, i][j\, k] + [l\, j][k\, i] + [l\, k][i\, j] = 0. \tag{6.40}$$

6.1.2 Polarization vectors

Finally, we have the polarization vectors,

$$\epsilon_-^\mu(p;q) = -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[q\, p]}, \quad \epsilon_+^\mu(p;q) = -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle q\, p\rangle}, \tag{6.41}$$

where q is a lightlike *reference momentum*. We can verify these expressions for a specific value of p, and then say that the general case follows due to Lorentz transformation properties of twistors and polarization vectors.

Choose a frame in which $p = (E, 0, 0, E)$. The most general form of the polarization vector for this momentum is

$$\epsilon_+(p) = \frac{e^{i\phi}}{\sqrt{2}}(0, 1, -i, 0) + C\,p, \tag{6.42}$$

where $e^{i\phi}$ is an arbitrary phase factor and C is a complex number. The freedom to add a multiple of p comes due to the fact that $p^2 = 0$. Using the explicit form of twistors,

$$|p\rangle = \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix}, |p] = \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix}, \quad \text{and} \quad |q\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix}, \tag{6.43}$$

where α, β are complex numbers. Proper contraction of spinor indices, $\langle q|\gamma^\mu|p] = \langle q|_{\dot{a}}\,\bar{\sigma}^{\mu\dot{a}a}|p]_a$, gives

$$\epsilon_+(p) = -\frac{\langle q|\bar{\sigma}|p]}{\sqrt{2}\langle q\,p\rangle} = \begin{pmatrix} \frac{-\alpha}{\sqrt{2}\beta} & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & \frac{-\alpha}{\sqrt{2}\beta} \end{pmatrix}, \tag{6.44}$$

so that the relation holds with $e^{i\phi} = 1$ and $C = -\beta/\sqrt{2}\alpha E$. This identity can be verified for the negative helicity polarization vector by using $\epsilon_+^{\mu *} = \epsilon_-^\mu$ and taking a complex conjugate.

6.2 Compton scattering with twistor variables

With these tools we are ready to compute some scattering amplitudes. Consider (massless) spinor electrodynamics,

$$L = i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + e\bar{\psi}A\!\!\!/\psi. \tag{6.45}$$

In terms of twistors, we have the following Feynman rules.

- Outgoing fermion with $h = +1/2$: $[p|$
- Outgoing fermion with $h = -1/2$: $\langle p|$
- Outgoing antifermion with $h = +1/2$: $|p]$
- Outgoing antifermion with $h = -1/2$: $|p\rangle$
- Outgoing photon with $h = \pm$: $\epsilon_\pm^\mu(p)$

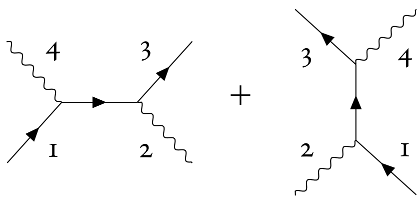
Due to crossing symmetry, we can treat an incoming fermion (antifermion) with helicity h as an outgoing antifermion (fermion) with helicity $-h$. Similarly, an incoming photon with helicity h can be treated as an outgoing photon with helicity $-h$.

In spinor electrodynamics, the polarization vector is always contracted with γ-matrices. We have the following useful forms,

$$\epsilon\!\!/_-(p;q) = \frac{\sqrt{2}}{[q\,p]}(|p\rangle[q| + |q]\langle p|), \quad \epsilon\!\!/_+(p;q) = \frac{\sqrt{2}}{\langle q\,p\rangle}(|p]\langle q| + |q\rangle[p|), \tag{6.46}$$

which are obtained by an application of the Fierz identities.

In $e\gamma \to e\gamma$ scattering, the following diagrams contribute

$A(e\gamma \to e\gamma) \quad =$  (6.47)

We will compute the above process for specific helicity assignments. There are a total of 16 different helicity assignments, but only 8 of these are independent; the other 8 are obtained by flipping all helicities.

Consider a process in which electrons have the opposite helicity. For example,

$$A(e^-\gamma \to e^+\gamma) = ie^2\frac{[3|\not{\epsilon}_2(-\not{p}_1+\not{p}_4)\not{\epsilon}_4|1]}{(p_1-p_4)^2} + ie^2\frac{[3|\not{\epsilon}_4(-\not{p}_1-\not{p}_2)\not{\epsilon}_2|1]}{(p_1+p_2)^2}. \tag{6.48}$$

But note that an odd number of gamma-matrices are sandwiched between spinors of the same type. Such spinor products vanish (because indices cannot be contracted to form a Lorentz scalar). Hence, processes with different initial and final electron helicities are forbidden.

Consider a process in which photons have the opposite helicity,

$$A(e^-\gamma^- \to e^-\gamma^+) = ie^2\frac{\langle 3|\not{\epsilon}_{2+}(-\not{p}_1+\not{p}_4)\not{\epsilon}_{4+}|1]}{(p_1-p_4)^2} + ie^2\frac{\langle 3|\not{\epsilon}_{4+}(-\not{p}_1-\not{p}_2)\not{\epsilon}_{2+}|1]}{(p_1+p_2)^2} \tag{6.49}$$

Numerator of the first term is proportional to $\langle 3\,q_2\rangle$ which can be made to vanish by choosing $q_2 = p_3$. Likewise, numerator of the second term is proportional to $\langle 3\,q_4\rangle$, and can be made to vanish by choosing $q_4 = p_3$. Hence, processes with different initial and final photon helicities are also forbidden.

The only remaining amplitudes are $A(e^-\gamma^- \to e^-\gamma^-)$, $A(e^-\gamma^+ \to e^-\gamma^+)$ and their crossing related cousins $A(e^+\gamma^+ \to e^+\gamma^+)$ and $A(e^+\gamma^- \to e^+\gamma^-)$.

We start with

$$\begin{aligned} &A(e^-\gamma^- \to e^-\gamma^-) \\ &= i(\sqrt{2}e)^2\left(\frac{\langle 3\,q_2\rangle[2\,1]\langle 1\,4\rangle[4\,1]}{\langle q_2\,2\rangle\langle 1\,4\rangle[4\,1][q_4\,4]} + \frac{\langle 3\,4\rangle[q_4|(|1]\langle 1| + |2]\langle 2|)|q_2\rangle[2\,1]}{[q_4\,4]\langle 1\,2\rangle[1\,2]\langle q_2\,2\rangle}\right) \end{aligned} \tag{6.50}$$

Choosing $q_2 = p_3$ and $q_4 = p_1$, the first term vanishes and we have,

$$A(e^-\gamma^- \to e^-\gamma^-) = i(\sqrt{2}e)^2\frac{\langle 3\,4\rangle[2\,1]}{[1\,4]\langle 1\,2\rangle} = i(\sqrt{2}e)^2\frac{[1\,2]^2}{[1\,4][4\,3]}. \tag{6.51}$$

And its crossing related cousin,

$$A(e^+\gamma^+ \to e^+\gamma^+) = i(\sqrt{2}e)^2\frac{\langle 1\,2\rangle^2}{\langle 1\,4\rangle\langle 4\,3\rangle}. \tag{6.52}$$

Proceeding similarly,

$$\begin{aligned} &A(e^-\gamma^+ \to e^-\gamma^+) \\ &= i(\sqrt{2}e)^2\left(\frac{\langle 3\,2\rangle[q_2|(|1]\langle 1| + |4]\langle 4|)|q_4\rangle[4\,1]}{[q_2\,2]\langle 1\,4\rangle[4\,1]\langle q_4\,4\rangle} + \frac{\langle 3\,q_4\rangle[4\,1]\langle 1\,2\rangle[q_2\,1]}{\langle q_4\,4\rangle[1\,2]\langle 1\,2\rangle[q_2\,2]}\right) \end{aligned} \tag{6.53}$$

Choosing $q_2 = p_1$ and $q_4 = p_3$, the second term vanishes and we have,

$$A(e^-\gamma^+ \to e^-\gamma^+) = i(\sqrt{2}e)^2\frac{\langle 2\,3\rangle[1\,4]}{[1\,2]\langle 1\,4\rangle} = i(\sqrt{2}e)^2\frac{[1\,4]^2}{[1\,2][2\,3]}. \tag{6.54}$$

And its crossing related cousin,

$$A(e^+\gamma^- \to e^+\gamma^-) = i(\sqrt{2}e)^2 \frac{\langle 14\rangle^2}{\langle 12\rangle\langle 23\rangle}. \tag{6.55}$$

For the unpolarized cross section of this process, we average over initial helicities and sum over final helicities,

$$\left\langle |A(e\gamma \to e\gamma)|^2 \right\rangle = \frac{8e^4}{4}\left(\frac{s_{12}^2}{s_{14}s_{34}} + \frac{s_{14}^2}{s_{12}s_{23}}\right) = 2e^4\left(\frac{s_{12}}{s_{14}} + \frac{s_{14}}{s_{12}}\right) \tag{6.56}$$

where $s_{ij} = -(p_i + p_j)^2 = -2p_i \cdot p_j$ (for lightlike momenta). For diagrams as drawn above, the Mandelstam variables are $s_{12} = s$, and $s_{14} = u$. For comparision, the spin-averaged amplitude squared for Compton scattering obtained from traditional methods is

$$\left\langle |A(e\gamma \to e\gamma)|^2 \right\rangle = 2e^4\left[\frac{m^4 + m^2(3s+u) - su}{(m^2 - s)^2} + \frac{m^4 + m^2(3u+s) - su}{(m^2 - u)^2} + \frac{2m^2(s+u+2m^2)}{(m^2 - s)(m^2 - u)}\right], \tag{6.57}$$

which in the limit $m^2/s, m^2/u \ll 1$,

$$\left\langle |A(e\gamma \to e\gamma)|^2 \right\rangle = 2e^4\left(\frac{s}{u} + \frac{u}{s}\right), \tag{6.58}$$

is identical to the spinor helicity result but computed with much greater effort.

It is remarkable that each amplitude is given only as a product of twistor brackets and nothing else. In a sense, twistor variables give a unified representation of different massless particles; instead of dealing with gamma matrices, Dirac spinors and polarization vectors separately, to compute amplitudes in spinor helicity formalism one only needs to deal with twistor variables.

Chapter 7
Amplitudes in Nonabelian Gauge Theory

Unlike electrodynamics, in nonabelian gauge theory vertex factors like,

$$i\mathbf{V}^{abc}_{\mu\nu\rho}(k_1,k_2,k_3) = gf^{abc}[(k_1-k_2)_\rho g_{\mu\nu} + (k_2-k_3)_\mu g_{\nu\rho} + (k_3-k_1)_\nu g_{\rho\mu}], \quad (7.1)$$

make calculation of even tree level processes extremely complicated. The reason for this increased complexity is twofold: (1) colour factors (2) products of gamma matrices, Dirac spinors, and polarization vectors. As seen with electrodynamics one can use twistor variables to reduce complexity that stems from (2). For dealing with colour factors, we introduce the Gervais–Neveu gauge and colour ordering.

Consider an SU(N) gauge theory described by the Yang–Mills lagrangian,

$$L_{\mathrm{YM}} = -\frac{1}{4}\operatorname{Tr} F^{\mu\nu}F_{\mu\nu}. \quad (7.2)$$

For computing amplitudes, it is convenient to work in the Gervais–Neveu gauge, which has the gauge fixing term

$$L_{\mathrm{gf}} = -\frac{1}{2}\operatorname{Tr}\left(H^{\mu}_{\mu}\right)^2, \quad (7.3)$$

where $H_{\mu\nu} = \partial_\mu A_\nu - \frac{ig}{\sqrt{2}}A_\mu A_\nu$. After gauge fixing, the lagrangian takes the following form,

$$L = \operatorname{Tr}\left(-\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - i\sqrt{2}g\,\partial^\mu A^\nu A_\nu A_\mu + \frac{1}{4}g^2 A^\mu A^\nu A_\mu A_\nu\right). \quad (7.4)$$

Let $A(1,\dots,n)$ denote the scattering amplitude with n external gluons, all gluons considered outgoing. Then the tree level amplitudes have the following color structure,

$$A(1,\dots,n) = g^{n-2}\sum_{\text{noncyclic perms}} \operatorname{Tr}(T^{a_1}\cdots T^{a_n})A[1,\dots,n], \quad (7.5)$$

where $A[1,\dots,n]$ is a *color-ordered* partial amplitude. We can read off the color ordered Feynman rules for vertices from Gervais–Neveu gauge fixed lagrangian,

- 3-gluon vertex $\mathbf{V}_{\mu\nu\rho}(p,q,r) = -\sqrt{2}(g_{\mu\nu}p_\rho + g_{\nu\rho}q_\mu + g_{\rho\mu}r_\nu)$,
- 4-gluon vertex $\mathbf{V}_{\mu\nu\rho\sigma} = g_{\mu\rho}g_{\nu\sigma}$.

These color ordered amplitudes have the following useful relations among themselves

1. Cyclicity: $A[1,2,\ldots,n]=A[2,\ldots,n,1]$, etc.
2. Reflection: $A[n,\ldots,2,1]=(-1)^n A[1,2,\ldots,n]$.
3. Decoupling of the fictitious photon:

$$A[1,2,3,\ldots,n]+A[2,1,3,\ldots,n]+A[2,3,1,\ldots,n]+\cdots+A[2,3,\ldots,1,n]=0. \tag{7.6}$$

In the following, we adopt a convention in which all particles are considered outgoing. In this convention, an incoming particle with helicity h, will become an outgoing antiparticle with helicity $-h$.

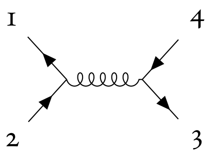
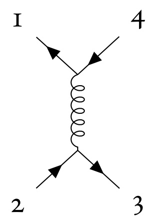
If we also include a family of quarks in the fundamental representation of the gauge group SU(N), the interaction Lagrangian is $L_1=(ig/\sqrt{2})\bar{\psi}\not{A}\psi$, and the color ordered vertex rule is,

– Quark-gluon interaction vertex = $\mathbf{V}^\mu=\frac{ig}{\sqrt{2}}\gamma^\mu$.

In general, colour factors for a process can be obtained by drawing double line versions of Feynman diagrams and contracting indices properly (cf. Appendix B).

7.1 $qq \to qq$

Let us start by computing an amplitude for $qq \to qq$ scattering (and its crossing related cousins). For the colour ordered amplitude, we have the following contributing diagram,

$iA[1_q,2_{\bar{q}},3_q,4_{\bar{q}}] =$  $+$  (7.7)

As with Compton scattering, we are going to compute this process for specific helicity assignments. In our experience with QED, the only nonvanishing diagrams are those in which helicities are "conserved", therefore the only nonvanishing amplitudes are $A[1^-_q,2^+_{\bar{q}},3^-_q,4^+_{\bar{q}}]$, $A[1^-_q,2^+_{\bar{q}},3^+_q,4^-_{\bar{q}}]$, $A[1^-_q,2^-_{\bar{q}},3^+_q,4^+_{\bar{q}}]$ and the other three amplitudes obtained by flipping all helicities.

We have,

$$\begin{aligned} A[1^-_q,2^+_{\bar{q}},3^-_q,4^+_{\bar{q}}] &= \frac{g^2}{2}\left(\frac{\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4]}{-s_{12}}+\frac{\langle 1|\gamma^\mu|3]\langle 2|\gamma_\mu|4]}{-s_{13}}\right) \\ &= g^2\left(\frac{\langle 13\rangle^2}{\langle 12\rangle\langle 43\rangle}+\frac{\langle 12\rangle^2}{\langle 13\rangle\langle 42\rangle}\right) \end{aligned} \tag{7.8}$$

using Fierz identity. Similarly,

$$A[1_q^-, 2_{\bar{q}}^+, 3_q^+, 4_{\bar{q}}^-] = \frac{g^2}{2} \frac{\langle 1|\gamma^\mu|2][3|\gamma_\mu|4\rangle}{-s_{12}} = g^2 \frac{\langle 14\rangle^2}{\langle 12\rangle\langle 34\rangle}, \tag{7.9}$$

and

$$A[1_q^-, 2_{\bar{q}}^-, 3_q^+, 4_{\bar{q}}^+] = \frac{g^2}{2} \frac{\langle 1|\gamma^\mu|4][3|\gamma_\mu|2\rangle}{-s_{14}} = g^2 \frac{\langle 12\rangle^2}{\langle 23\rangle\langle 41\rangle}. \tag{7.10}$$

As with QED the other two amplitudes are related by complex conjugation.

7.2 $q\bar{q} \to gg$

Before computing this amplitude with external gluons, note the following

$$\epsilon_+(p;q) \cdot \epsilon_+(p';q') = \frac{\langle q\, q'\rangle [p\, p']}{\langle q\, p\rangle \langle q'\, p'\rangle}, \tag{7.11}$$

$$\epsilon_-(p;q) \cdot \epsilon_-(p';q') = \frac{[q\, q']\langle p\, p'\rangle}{[q\, p][q'\, p']}, \tag{7.12}$$

$$\epsilon_+(p;q) \cdot \epsilon_-(p';q') = \frac{\langle q\, p'\rangle [p\, q']}{\langle q\, p\rangle [q'\, p']}, \tag{7.13}$$

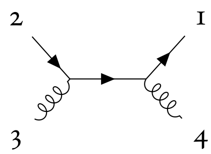
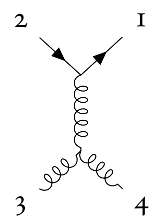
which are obtained using the form of polarization vectors given in the last chapter and Fierz identities. Dotting with momenta, we also have

$$k \cdot \epsilon_+(p;q) = \frac{\langle q\, k\rangle [k\, p]}{\sqrt{2}\langle q\, p\rangle}, \tag{7.14}$$

$$k \cdot \epsilon_-(p;q) = \frac{[q\, k]\langle k\, p\rangle}{\sqrt{2}[q\, p]}. \tag{7.15}$$

Note that $p \cdot \epsilon(p;q) = 0$ and $q \cdot \epsilon(p;q) = 0$, as expected for polarization vectors.

Next, let us consider an annihilation process $q\bar{q} \to gg$ (and its crossing related cousins). We have the following diagrams for the colour ordered amplitude.

$A[1_q, 2_{\bar{q}}, 3, 4] =$  $+$  (7.16)

As before, we are going to compute the amplitude for specific helicity assignments. Again, due to our experience with QED we expect the amplitude to vanish unless the quarks have opposite helicity. It can also be shown with a clever choice of reference vectors that the amplitude vanishes unless gluons have opposite helicity.

The only nonzero amplitudes are $A[1_q^-, 2_{\bar{q}}^+, 3^-, 4^+]$, $A[1_q^-, 2_{\bar{q}}^+, 3^+, 4^-]$ and two other obtained by flipping all helicities.

Consider,

$$A[1_q^-, 2_{\bar{q}}^+, 3^-, 4^+] = \frac{g^2}{2} \frac{\langle 1|\not\epsilon_{4+}(-\not p_1 - \not p_4)\not\epsilon_{3-}|2]}{-s_{14}} - g^2 \frac{\langle 1|\gamma_\sigma|2]\epsilon_{3\mu}\epsilon_{4\nu}\mathbf{V}_{345}^{\mu\nu\sigma}}{-s_{12}}. \quad (7.17)$$

Choosing $q_3 = p_4$ and $q_4 = p_3$, we have

$$\epsilon_{3\mu}\epsilon_{4\nu}\mathbf{V}_{345}^{\mu\nu\sigma} = (\epsilon_{3-} \cdot \epsilon_{4+})p_3^\sigma + (\epsilon_{3-} \cdot p_4)\epsilon_{4+}^\sigma + (\epsilon_{4+} \cdot (p_3 + p_4))\epsilon_{3-}^\sigma. \quad (7.18)$$

With above choices of reference momenta, $\epsilon_3(p_3; p_4) \cdot p_4 = 0$, and $\epsilon_4(p_4; p_3) \cdot (p_3 + p_4) = 0$. Using the polarization product identities, the first term is proportional to $\langle 3\,3\rangle[4\,4] = 0$. Therefore, the second diagram does not contribute.

From the first diagram, we have

$$A[1_q^-, 2_{\bar{q}}^+, 3^-, 4^+] = g^2 \frac{\langle 1\,3\rangle[4\,1]\langle 1\,3\rangle[4\,2]}{\langle 3\,4\rangle[4\,3]\langle 4\,1\rangle[4\,1]} = g^2 \frac{\langle 1\,3\rangle^3}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,1\rangle}. \quad (7.19)$$

Similarly, only the first diagram contributes to the amplitude $A[1_q^-, 2_{\bar{q}}^+, 3^+, 4^-]$. We have

$$A[1_q^-, 2_{\bar{q}}^+, 3^+, 4^-] = g^2 \frac{[2\,3]\langle 4\,1\rangle[1\,3]\langle 4\,1\rangle}{\langle 4\,3\rangle[3\,4]\langle 1\,4\rangle[1\,4]} = \frac{\langle 1\,4\rangle^3\langle 2\,4\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}. \quad (7.20)$$

The remaining nonvanishing amplitudes are related by complex conjugation.

7.3 $gg \to gg$

Before computing this amplitude note the following: If all of gluon helicities are the same or all but one of the helicities are the same then the corresponding amplitude vanishes at tree level.

We can go about proving the above claim as follows.

Consider $A[1^\pm, 2^+, 3^+, \ldots, n^+]$. In a tree diagram with n external gluons, there are going to be n polarization vectors, one from each external gluon. Moreover at tree level there are no more than $n-2$ three point gluon vertices. In each term of the partial amplitude, the polarization vector should either be contracted with a momentum factor from a 3-gluon vertex or another polarization vector. As the number of polarization vectors is n, while the number of 3-gluon vertices is $n-2$, there must be at least one product of polarization vectors in each term of the amplitude.

If we make a clever choice of reference momenta, $q_1 = q_2 = \cdots = q_n = p_1$, all of the polarization products vanish, and therefore the amplitude is zero.

Now, back to $gg \to gg$ scattering. Due to the above argument, the only nonvanishing four gluon amplitudes are those in which two gluons have positive helicity and two have negative helicity. Starting with $A[1^-, 2^-, 3^+, 4^+]$, we have the diagrams in Fig. 7.1.

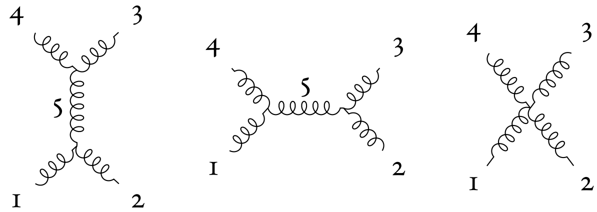


Figure 7.1: Contributing tree level diagrams for $gg \to gg$ process.

We choose the reference momenta to make most of the polarization products vanish. Choosing $q_1 = q_2 = p_4$, we have

$$\epsilon_{1-} \cdot \epsilon_{2-} = 0, \quad \epsilon_{1-} \cdot \epsilon_{4+} = 0, \quad \epsilon_{2-} \cdot \epsilon_{4+} = 0,$$

and choosing $q_3 = q_4 = p_1$, we have

$$\epsilon_{1-} \cdot \epsilon_{3+} = 0, \quad \epsilon_{3+} \cdot \epsilon_{4+} = 0. \tag{7.21}$$

The only nonvanishing polarization product is

$$\epsilon_{2-} \cdot \epsilon_{3+} = \frac{\langle q_3 2\rangle[3 q_2]}{\langle q_3 3\rangle[q_2 2]} = \frac{\langle 12\rangle[34]}{\langle 13\rangle[42]}. \tag{7.22}$$

We calculate vertex factors for the first diagram. With $p_5 = -p_1 - p_2$, the 125 vertex is

$$\begin{aligned}
\mathbf{V}^{\mu}_{125} &= -\sqrt{2}\big[(\epsilon_1 \cdot \epsilon_2) p_1^{\mu} + (p_2 \cdot \epsilon_1)\epsilon_2^{\mu} + (p_5 \cdot \epsilon_2)\epsilon_1^{\mu}\big] \\
&= -\sqrt{2}\big[(p_2 \cdot \epsilon_1)\epsilon_2^{\mu} + (p_5 \cdot \epsilon_2)\epsilon_1^{\mu}\big] \\
&= -\sqrt{2}\big[(p_2 \cdot \epsilon_1)\epsilon_2^{\mu} - (p_1 \cdot \epsilon_2)\epsilon_1^{\mu}\big],
\end{aligned} \tag{7.23}$$

where first term vanishes because $\epsilon_1 \cdot \epsilon_2 = 0$. Similarly, the 345 vertex is

$$\begin{aligned}
\mathbf{V}^{\nu}_{345} &= -\sqrt{2}[(\epsilon_3 \cdot \epsilon_4) p_3^{\nu} + (p_4 \cdot \epsilon_3)\epsilon_4^{\nu} + (-p_5 \cdot \epsilon_4)\epsilon_3^{\nu}] \\
&= -\sqrt{2}[(p_4 \cdot \epsilon_3)\epsilon_4^{\nu} - (p_3 \cdot \epsilon_4)\epsilon_3^{\nu}].
\end{aligned} \tag{7.24}$$

Putting both of these pieces together, we have

$$i\frac{\mathbf{V}^{\mu}_{125} g_{\mu\nu} \mathbf{V}^{\nu}_{345}}{-s_{12}} = 2i\frac{(p_2 \cdot \epsilon_1)(p_3 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3)}{s_{12}}, \tag{7.25}$$

because all other polarization products vanish.

In the second diagram consider the 145 vertex; it is proportional to

$$(\epsilon_1 \cdot \epsilon_4) p_1 + (\epsilon_1 \cdot p_4)\epsilon_4 + (\epsilon_4 \cdot p_5)\epsilon_1. \tag{7.26}$$

The first term vanishes because $\epsilon_1 \cdot \epsilon_4 = 0$, the second term vanishes because $\epsilon_1 \cdot p_4 = \epsilon_1 \cdot q_1 = 0$, the third term vanishes because

$$\begin{aligned}\epsilon_4 \cdot p_5 &= \epsilon_4 \cdot (-p_1 - p_4) \\ &= \epsilon_4 \cdot (-q_4 - p_4) = 0.\end{aligned} \tag{7.27}$$

Therefore the second diagram does not contribute.

Finally, the third diagram, with the four point contact vertex, contains the terms $\epsilon_1 \cdot \epsilon_3$ and $\epsilon_2 \cdot \epsilon_4$, which are both zero. Hence, it doesn't contribute either.

The only contribution to this process comes from the first diagram,

$$\begin{aligned}A[1^-,2^-,3^+,4^+] &= 2\frac{(p_2 \cdot \epsilon_1)(p_3 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3)}{-s_{12}} \\ &= \frac{[4\,2]\langle 2\,1\rangle\langle 1\,3\rangle[3\,4]\langle 1\,2\rangle[3\,4]}{\langle 1\,2\rangle[2\,1][4\,1]\langle 1\,4\rangle\langle 1\,3\rangle[4\,2]} \\ &= \frac{\langle 2\,1\rangle[3\,4]^2}{[2\,1][4\,1]\langle 1\,4\rangle}\end{aligned} \tag{7.28}$$

Multiplying the above expression by $\langle 2\,1\rangle/\langle 2\,1\rangle$, using $\langle 2\,1\rangle[2\,1] = \langle 3\,4\rangle[3\,4]$, then multiplying by $\langle 4\,1\rangle/\langle 4\,1\rangle$ and using $[3\,4]\langle 4\,1\rangle = -[3\,2]\langle 2\,1\rangle$, we have

$$A[1^-,2^-,3^+,4^+] = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}. \tag{7.29}$$

Using cyclic property of color-ordered amplitudes, we can obtain all other amplitudes in which the negative helicity gluons are adjacent from the above amplitude. For example

$$A[1^+,2^-,3^-,4^+] = \frac{\langle 2\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}, \quad A[1^-,2^+,3^+,4^-] = \frac{\langle 1\,4\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}. \tag{7.30}$$

Using U(1) decoupling $A[1,2,3,4] + A[2,1,3,4] + A[2,3,1,4] = 0$,

$$A[1^-,2^+,3^-,4^+] = -A[2^+,1^-,3^-,4^+] - A[2^+,3^-,1^-,4^+], \tag{7.31}$$

and simplifying using the Schouten identity, we can get amplitudes in which negative helicity gluons are not adjacent,

$$A[1^-,2^+,3^-,4^+] = \frac{\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}. \tag{7.32}$$

These are examples of the famous Parke–Taylor amplitudes for $n = 4$. An inductive proof of the general Parke–Taylor formula for any n will be given in the next chapter.

7.4 Three particle special kinematics

Despite appearing simpler than the four point amplitude, there are some kinematic considerations for three point gluon amplitudes due to which we have to be careful when choosing reference momenta in polarization vectors. As a result we cannot make "clever choices" as we have been doing till now to simplify amplitudes.

The first thing to note is that, for real valued momenta the all three point gluon amplitudes vanish due to arguments at the beginning of the last section. However, in the next chapter we are going to introduce complex shifts of momentum, so a *shifted* 3-point amplitude would not in general be zero.

Consider a scattering process of three massless particles with all particles outgoing, having momenta p, q and r. Momentum vectors are lightlike, i.e., $p^2 = q^2 = r^2 = 0$, and satisfy momentum conservation $p + q + r = 0$. We have

$$\langle p\, q\rangle[p\, q] = 2p\cdot q = (p+q)^2 = r^2 = 0, \tag{7.33}$$

which means either $\langle p\, q\rangle = 0$ or $[p\, q] = 0$[1]. Let $\langle p\, q\rangle = 0$ and $[p\, q] \neq 0$; then we have

$$[p\, q]\langle q\, r\rangle = -[p|\not{q}|r\rangle = -[p|(\not{p} + \not{r})|r\rangle = 0, \tag{7.34}$$

therefore $\langle q\, r\rangle = 0$. Similarly, one can also show $\langle p\, r\rangle = 0$. Hence we have $\langle p\, q\rangle = \langle q\, r\rangle = \langle r\, p\rangle = 0$, or alternatively

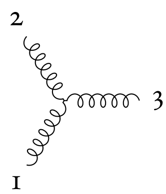
$$|p\rangle \propto |q\rangle \propto |r\rangle\,. \tag{7.35}$$

Had we taken $[p\, q] = 0$ and $\langle p\, q\rangle \neq 0$, we would have concluded $[p\, q] = [q\, r] = [r\, p] = 0$ and

$$|p] \propto |q] \propto |r]. \tag{7.36}$$

For three point amplitudes with external gauge bosons, one must take care when choosing reference momenta for polarization vectors and make sure that the choice does not lead to a spinor bracket vanishing in the denominator.

Let us now consider the color-ordered three point gluon amplitude,



$$A[1^-, 2^-, 3^+] \\ = -\frac{[q_1\, q_2]\langle 1\, 2\rangle\langle q_3\, 2\rangle[2\, 3] + \langle q_3\, 2\rangle[3\, q_2][q_1\, 2]\langle 2\, 1\rangle + \langle q_3\, 1\rangle[3\, q_1][q_2\, 3]\langle 3\, 2\rangle}{[q_1\, 1][q_2\, 2]\langle q_3\, 3\rangle}. \tag{7.37}$$

[1]If momenta were real valued we would have $\langle p\, q\rangle^* = [q\, p]$ and therefore $\langle p\, q\rangle = 0 \iff [p\, q] = 0$.

Due to three particle kinematics, we cannot make a clever choice of reference momenta q_j to simplify the amplitude without making the denominator zero. If we have $|1\rangle \propto |2\rangle \propto |3\rangle$, each term in the numerator vanishes and the amplitude becomes zero, so we choose $|1] \propto |2] \propto |3]$. With this choice, the first term vanishes and,

$$A[1^-,2^-,3^+] = -\frac{\langle q_3\,2\rangle[3\,q_2][q_1\,2]\langle 2\,1\rangle + \langle q_3\,1\rangle[3\,q_1][q_2\,3]\langle 3\,2\rangle}{[q_1\,1][q_2\,2]\langle q_3\,3\rangle}.$$

Using momentum conservation relations and the Schouten identity, the amplitude can be simplified to

$$A[1^-,2^-,3^+] = \frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 3\,1\rangle} = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle}. \tag{7.38}$$

Similarly, we can also compute the amplitude in which two helicities are positive and one is negative,

$$A[1^+,2^+,3^-] = \frac{[1\,2]^3}{[2\,3][3\,1]} = \frac{[1\,2]^4}{[1\,2][2\,3][3\,4]}. \tag{7.39}$$

Remarkably, these amplitudes have a form very similar to what we obtained for four point gluon amplitudes. As we shall see, this is not a coincidence: these are both special cases of the general n-point MHV amplitude.

CHAPTER 8

Complex Shifts, Recursion Relations, and the Parke–Taylor Formula

In this chapter we are going to give an inductive proof of the Parke–Taylor formula for n-gluon MHV amplitude. The base case for this induction proof is the three point amplitude $A[1,2,3]$, which we computed in the last section of the previous chapter. In the next section we start with a general description of complex shifts and on-shell recursion relations before specializing to the BCFW shift. Finally, we prove the Parke–Taylor formula using BCFW recursion.

8.1 On-shell recursion relations

We are now going to set up the machinery of *on-shell recursion relations* for computing tree level amplitudes. For an n-point on-shell amplitude, introduce n shift vectors r_i, which could, in general, be complex valued. We require r_i to satisfy the following

1. Momentum conservation: $\sum_{i=1}^{n} r_i = 0$,
2. Orthogonality: $r_i \cdot r_j = 0$, and in particular $r_i^2 = 0$ for each i,
3. $p_i \cdot r_i = 0$ (no sum) for each i.

With these, we define *shifted momenta*

$$\hat{p}_i = p_i + z r_i, \tag{8.1}$$

where $z \in \mathbb{C}$. By the requirements (1)–(3) on r_i, the shifted momenta satisfy momentum conservation $\sum_{i=1}^{n} \hat{p}_i = 0$, and the on shell condition $\hat{p}_i^2 = 0$.

In general, an amplitude will have Feynman propagators for every internal line. An internal line carrying momentum $P_I = \sum_{i \in I} p_i$—where I indexes a collection of momenta—will bring a factor of $1/P_I^2$ due to the Feynman propagator. With shifted momenta we define $\hat{P}_I = \sum_{i \in I} \hat{p}_i$, and we have

$$\hat{P}_I^2 = (P_I + zR_I)^2 = P_I^2 + 2zP_I \cdot R_I, \tag{8.2}$$

where $R_I = \sum_{i \in I} r_i$, and the z^2 term vanishes due to property (2) of the shift vectors. We can pull out various factors to write

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I) \quad \text{with} \quad z_I = -\frac{P_I^2}{2P_I \cdot R_I}. \tag{8.3}$$

As the shifted momenta satisfy momentum conservation and the on-shell condition, we can define a *shifted on-shell amplitude* $\hat{A}_n(z)$ with $p_i \to \hat{p}_i$. Due to arguments of the last paragraph, each Feynman propagator carrying momentum $P_I \to \hat{P}_I$, will bring a simple pole at $z = z_I$ into the shifted amplitude. If we are interested only in tree level processes, the amplitude does not have any branch cuts or other singularities; simple poles coming from Feynman propagators are the only singularities.

The shifted amplitude as defined above is a function of z in the complex plane. In fact, for tree level amplitudes, $\hat{A}_n(z)$ is a rational function of z and is therefore holomorphic everywhere except at its poles.

Consider the function $\hat{A}(z)/z$ defined on the complex plane; it has a simple pole at $z = 0$ with residue $A_n = \hat{A}_n(z = 0)$, the unshifted amplitude. If we draw a contour in the complex plane containing all of its simple poles, including the one at the origin, we can integrate over this contour and use the residue theorem to write

$$A_n + \sum_{z_I} \text{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z} = B_n, \tag{8.4}$$

where the sum is over all poles, and B_n is the residue "at infinity". When $B_n = 0$, the recursion works and we have the n-point amplitude

$$A_n = -\sum_{z_I} \text{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z}, \tag{8.5}$$

as a sum of residues.

As discussed above, for tree level processes, poles occur only when a shifted propagator goes on-shell at z_I, i.e., $\hat{P}_I^2(z = z_I) = 0$. When a propagator goes on-shell near z_I, it corresponds to the exchange of a real particle, and as a result the total amplitude can be written as a product of two lower point on-shell subamplitudes,

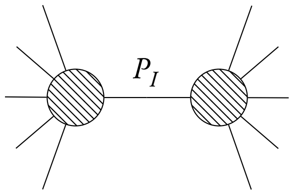
$$\hat{A}_n(z) \overset{\text{near } z_I}{\sim} \hat{A}_L(z_I)\frac{1}{\hat{P}_I}\hat{A}_R(z_I) = -\frac{z_I}{z - z_I}\hat{A}_L(z_I)\frac{1}{\hat{P}_I}\hat{A}_R(z_I). \tag{8.6}$$

With this it is easy to compute the residue of $\hat{A}_n(z)/z$ at z_I. We have,

$$-\text{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z} = \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I). \tag{8.7}$$

Therefore the unshifted on-shell amplitude is given by

$$A_n = \sum_I \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I) = \sum_I \tag{8.8}$$



where the sum is over all factorization channels I. Hence, an n-point on-shell amplitude is given as a product of lower point on-shell amplitudes. This forms the basis of the BCFW recursion relation that we will introduce in the next section.

8.2 Britto–Cachazo–Feng–Witten

In this section we are going to introduce a very special kind of shift called the BCFW (Britto–Cachazo–Feng–Witten) shift to derive BCFW recursion relations and finally prove the n-point Parke–Taylor amplitude formula.

Given an n-point amplitude with external (on-shell) momenta $p_1, \ldots, p_n$, pick out two of these momenta, say p_i and p_j $(i < j)$ and define shifts as follows,

$$|\hat{i}] = |i] + z|j], \quad |\hat{i}\rangle = |i\rangle, \quad |\hat{j}] = |j], \quad |\hat{j}\rangle = |j\rangle - z\,|i\rangle. \tag{8.9}$$

This is called an $[i,j\rangle$ shift. Using $-2p^\mu = \langle p|\gamma^\mu|p]$, we can write the shifts in terms of momenta,

$$\hat{p}_i^\mu = p_i^\mu - \frac{z}{2}\langle i|\gamma^\mu|j] \quad \text{and} \quad \hat{p}_j^\mu = p_j^\mu + \frac{z}{2}\langle i|\gamma^\mu|j], \tag{8.10}$$

so that the shift vectors are,

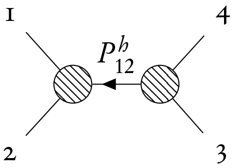
$$r_i^\mu = -\frac{1}{2}\langle i|\gamma^\mu|j], \quad r_j^\mu = +\frac{1}{2}\langle i|\gamma^\mu|j], \quad r_k^\mu = 0 \text{ when } k \neq i \text{ or } j. \tag{8.11}$$

An application of Fierz identities shows that these vectors satisfy the properties (1)–(3) required of shift vectors.

For Yang–Mills theory, in particular, the BCFW shift satisfies $\lim_{z\to\infty}\hat{A}_n(z) = 0$, when, in terms of helicities, the $[i,j\rangle$ shift looks like one of $[-,-\rangle$, $[-,+\rangle$ or $[+,+\rangle$. If the above holds, $B_n = 0$, and the recursion going to work.

Before going on to prove the Parke–Taylor formula, let us compute the four point gluon scattering amplitude $A[1^-, 2^-, 3^+, 4^+]$ using a $[1,2\rangle$ shift.

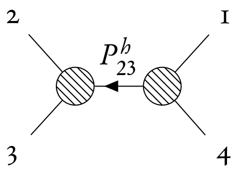
The first thing to note is that each subamplitude has to have at least three external points. Secondly, an internal propagator can go on shell only when 1 and 2 are in different subamplitudes. For if 1 and 2 are in the same subamplitude (cf. Figure below), the propagator carries momentum $P_{12} \to \hat{P}_{12} = \hat{p}_1 + \hat{p}_2 = p_1 + p_2 = P_{12}$, which does not go on-shell for generic momenta.



(8.12)

Finally, as we are calculating the color-ordered amplitude, we only draw diagrams in which gluons are labelled cyclically. With all these things in mind, there is only one factorization channel (two if we account for helicities) for the four point amplitude with a $[1,2\rangle$ shift.

$$A[1^-,2^-,3^+,4^+] \quad = \quad \sum_{h=\pm}$$



(8.13)

Using the recursion formula,

$$A[1^-,2^-,3^+,4^+] = \frac{\hat{A}[\hat{1}^-,\hat{P}_{23}^-,4^+]\,\hat{A}[\hat{2}^-,3^+,-\hat{P}_{23}^+]}{P_{23}^2} + \frac{\hat{A}[\hat{1}^-,\hat{P}_{23}^+,4^+]\,\hat{A}[\hat{2}^-,3^+,-\hat{P}_{23}^-]}{P_{23}^2} \tag{8.14}$$

Consider the three point amplitude,

$$\hat{A}[\hat{1}^-,\hat{P}_{23}^+,4^+] = \frac{[\hat{P}_{23}^+\,4]^3}{[\hat{1}\,\hat{P}_{23}][4\,\hat{1}]} \tag{8.15}$$

The shifted momentum is on-shell, i.e.,

$$\hat{P}_{23}^2 = (\hat{p}_2 + p_3)^2 = (\hat{p}_1 + p_4)^2 = [\hat{1}\,4]\langle\hat{1}\,4\rangle = [\hat{1}\,4]\langle 1\,4\rangle = 0. \tag{8.16}$$

For generic momenta this is only possible if $[\hat{1}\,4] = 0$. In the numerator we have

$$|\hat{P}_{23}\rangle\,[\hat{P}_{23}\,4] = -\hat{P}_{23}|4] = -(\hat{p}_2 + p_3)|4] = (\hat{p}_1 + p_4)|4] = \hat{p}_1|4] = |\hat{1}\rangle\,[\hat{1}\,4] = 0, \tag{8.17}$$

which, again, for generic momenta is only possible when $[\hat{P}_{23}\,4] = 0$. Similarly, we can also show $[\hat{1}\,\hat{P}_{23}] = 0$. As $\hat{P}_{23}$ goes on-shell, there are three powers of zero in the numerator and only two in the denominator, therefore, this amplitude vanishes.

Only the first term contributes to the four point amplitude. Using $P_{23}^2 = \langle 2\,3\rangle[2\,3]$ and the form of three point amplitudes, we have

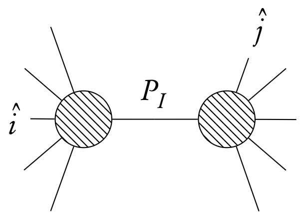
$$\begin{aligned} A[1^-,2^-,3^+,4^+] &= -\frac{\langle \hat{1}\,\hat{P}_{23}\rangle^3[3\,\hat{P}_{23}]^3}{\langle \hat{P}_{23}\,4\rangle\langle 4\,\hat{1}\rangle\langle 2\,3\rangle[2\,3][\hat{2}\,3][\hat{P}_{23}\,2]} \\ &= \frac{\langle 1\,2\rangle^3[2\,3]^3}{\langle 4\,1\rangle[2\,3]\langle 2\,3\rangle[2\,3]\langle 3\,4\rangle[2\,3]} \\ &= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}, \end{aligned} \tag{8.18}$$

where to simplify, we used $\langle \hat{1}\,\hat{P}_{23}\rangle[\hat{P}_{23}\,3] = -\langle 1\,2\rangle[2\,3]$ and $\langle \hat{P}_{23}\,4\rangle[\hat{P}_{23}\,\hat{2}] = -\langle 3\,4\rangle[2\,3]$.

For an amplitude in which the negative helicity gluons are not adjacent, like $A[1^-,2^+,3^-,4^+]$, the calculation is exactly the same, but instead of a $[1,2\rangle$ shift, one has to start with a $[1,3\rangle$ shift.

Adapting the general recursion relation for a BCFW $[i,j\rangle$ shift, we have the BCFW recursion formula,

$$A_n = \sum_I \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I) = \sum_I \tag{8.19}$$



where the sum is over all factorization channels such that p_i and p_j are on different subamplitudes. For otherwise, as in the $n=4$ case, the momentum $\hat{P}_I$ does not go on-shell.

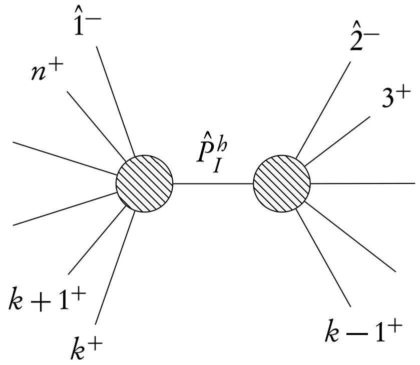
8.3 Proof of the Parke–Taylor formula

Computing an n-point MHV amplitude for gluons proceeds very similarly to the computation of the four point amplitude. For reference, the n-point Parke–Taylor formula is

$$A[1^-,2^-,3^+,\ldots,n^+] = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\cdots\langle n\,1\rangle}. \tag{8.20}$$

The proof will proceed by induction on the number of gluons n. We already have the result for 3 point amplitudes, which is going to serve as the base case of induction. As induction hypothesis assume that all lower point amplitudes are given by the Parke–Taylor formula. To compute the n point amplitude $A[1^-,2^-,3^+\ldots,n^+]$ recursively, we are going to use a $[1,2\rangle$ shift. According to BCFW recursion, the amplitude is

$$A[1^-,2^-,3^+,\dots,n^+] = \sum_{k=4}^{n}\sum_{h=\pm} \tag{8.21}$$



where the sum includes all factorization channels and there is also a sum over possible helicities of the exchanged gluon. In terms of subamplitudes, we have

$$\begin{aligned} &A[1^-,2^-,3^+,\dots,n^+] \\ &= \sum_{k=4}^{n}\sum_{h=\pm}\left(\hat{A}_L[\hat{1}^-,\hat{P}_I^h,k^+,\dots,n^+]\frac{1}{P_I^2}\hat{A}_R[-\hat{P}_I^{-h},\hat{2}^-,3^+,\dots,(k-1)^+]\right). \end{aligned} \tag{8.22}$$

Now, using the fact that all gluon tree amplitudes vanish, except $n=3$, in which one of the gluons has a helicity different from the rest vanishes, only two diagrams survive

$$\begin{aligned} A[1^-,2^-,3^+,\dots,n^+] &= \hat{A}_L[\hat{1}^-,-\hat{P}_{1n}^+,n^+]\frac{1}{P_{1n}^2}\hat{A}_R[\hat{P}_{1n}^-,\hat{2}^-,3^+,\dots,(n-1)^+] \\ &+\hat{A}_L[\hat{1}^-,\hat{P}_{23}^-,4^+,\dots,n^+]\frac{1}{P_{23}^2}\hat{A}_R[-\hat{P}_{23}^+,\hat{2}^-,3^+] \end{aligned} \tag{8.23}$$

As in the case of the four point amplitude,

$$\hat{A}[\hat{1}^-,-\hat{P}_{1n}^+,\hat{n}] = \frac{[\hat{P}_{1n}\,n]^3}{[n\,\hat{1}][\hat{1}\,\hat{P}_{1n}]} = 0, \tag{8.24}$$

because

$$\hat{P}_{1n}^2 = (\hat{p}_1+p_n)^2 = [\hat{1}\,n]\langle\hat{1}\,n\rangle = [\hat{1}\,n]\langle 1\,n\rangle = 0, \tag{8.25}$$

which can only occur for generic momenta if $[\hat{1}\,n]=0$. In the numerator,

$$|\hat{P}_{1n}\rangle[\hat{P}_{1n}\,n] = -\hat{P}_{1n}|n] = -(\hat{p}_1+p_n)|n] = -\hat{p}_1|n] = |\hat{1}\rangle[\hat{1}\,n] = 0, \tag{8.26}$$

which can only occur for generic momenta if $[\hat{P}_{1n}\,n]=0$. Similarly,

$$|\hat{P}_{1n}\rangle[\hat{1}\,\hat{P}_{1n}] = |n\rangle[\hat{1}\,n] = 0, \tag{8.27}$$

and therefore $[\hat{1}\,\hat{P}_{1n}]=0$. There are three powers of zero in the numerator and two powers in the denominator, hence as $\hat{P}_{1n}$ goes on-shell, this three point amplitude vanishes.

Now, we are only left with

$$A[1^-,2^-,3^+,\ldots,n^+]=\hat{A}_L[\hat{1}^-,\hat{P}_{23}^-,4^+,\ldots,n^+]\frac{1}{P_{23}^2}\hat{A}_R[-\hat{P}_{23}^+,\hat{2}^-,3^+]. \tag{8.28}$$

$\hat{A}_L$ is a $n-1$ point amplitude and A_R is a three point amplitude. By induction hypothesis,

$$\hat{A}_L[\hat{1}^-,\hat{P}_{23}^-,4^+,\ldots,n^+]=\frac{\langle\hat{1}\,\hat{P}_{23}\rangle^4}{\langle\hat{1}\,\hat{P}_{23}\rangle\langle\hat{P}_{23}\,4\rangle\langle 4\,5\rangle\cdots\langle n\,1\rangle}, \tag{8.29}$$

and due to the induction base case,

$$A_R[-\hat{P}_{23}^+,\hat{2}^-,3^+]=\frac{[\hat{P}_{23}\,3]^3}{[\hat{P}_{23}\,\hat{2}][\hat{2}\,3]}. \tag{8.30}$$

Using $P_{23}^2=\langle 2\,3\rangle[2\,3]$, and the relationships $\langle\hat{1}\,\hat{P}_{23}\rangle[\hat{P}_{23}\,3]=-\langle 1\,2\rangle[2\,3]$ and $\langle\hat{P}_{23}\,4\rangle[\hat{P}_{23}\,\hat{2}]=-\langle 3\,4\rangle[2\,3]$, we can simplify the n point amplitude to

$$A[1^-,2^-,3^+,\ldots,n^+]=\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\cdots\langle n\,1\rangle}. \tag{8.31}$$

This concludes the proof of Parke–Taylor formula.

For a version of the formula in which the negative helicity gluons are not adjacent, i.e., an amplitude like $A[1^-,2^+,\ldots,i^-,\ldots,n^+]$, the proof proceeds exactly as above but instead of a $[1,2\rangle$ shift, one has to start with a $[1,i\rangle$ shift to build appropriate recursion relations.

Bibliography

1. Mark Srednicki, Quantum Field Theory (Cambridge University Press, 2007)
2. Heniette Elvang, Yu-Tin Huang, Scattering Amplitudes in Gauge Theory and Gravity (Cambridge University Press, 2015)
3. Michael E. Peskin, Daniel V. Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley, 1995)
4. Gerard 't Hooft, The Conceptual Basis of Quantum Field Theory in Philosophy of Physics (North-Holland, 2007)
5. Stephen J. Parke, T. R. Taylor, Amplitude for n-Gluon Scattering, Physical Review Letters, 56, 2459 (1986)
6. F. A. Berends, W. T. Giele, Recursive Calculations for Processes with n Gluons, Nuclear Physics B, 306 (4): 759–808 (1988)
7. Ruth Britto, Freddy Cachazo, Bo Feng, Edward Witten, Direct Proof of the Tree-Level Scattering Amplitude Recursion Relation in Yang–Mills Theory, Physical Review Letters, 94, 181602 (2005)

Appendix A

Reference Formulae

Some formulae that are used throughout the main text are collected here for quick reference.

Minkowski metric has the mostly positive signature: $g_{\mu\nu} = \text{diag}(-,+,+,+)$. We define two-dimensional generators of Lorentz group,

$$\sigma^{\mu}_{a\dot{b}} = (1,\sigma), \quad \bar{\sigma}^{\mu\dot{a}b} = (1,-\sigma), \tag{A.1}$$

where σ are Pauli matrices. Two component spinor indices are raise and lowered using

$$\epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{ab} = -\epsilon_{\dot{a}\dot{b}} \tag{A.2}$$

A.1 Integrals

Feynman's trick to convert a reciprocal of products into an integral:

$$\frac{1}{A_1 \dots A_n} = \int dF_n (x_1 A_1 + \dots + x_n A_n)^{-n}, \tag{A.3}$$

where

$$\int dF_n = (n-1)! \int_0^1 dx_1 \dots dx_n \delta(x_1 + \dots + x_n - 1). \tag{A.4}$$

For just two factors it reduces to:

$$\frac{1}{A_1 A_2} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2} \tag{A.5}$$

Symmetric integration identity:

$$\int \frac{d^d k}{(2\pi)^d} k_\mu k_\nu f(k^2) = d^{-1} g_{\mu\nu} \int \frac{d^d k}{(2\pi)^d} k^2 f(k^2). \tag{A.6}$$

For d-dimensional Euclidean space integrals:

$$\int \frac{d^d\bar{k}}{(2\pi)^d}\frac{(\bar{k}^2)^a}{(\bar{k}^2+D)^b} = \frac{\Gamma(b-a-\frac{d}{2})\Gamma(a+\frac{d}{2})}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac{d}{2})}D^{-(b-a-d/2)} \tag{A.7}$$

Wick rotation to convert a Minkowski space integral to a Euclidean space integral: $\bar{k}_j = k_j$ for $j = 1,\ldots,d-1$, and $\bar{k}_d = ik^0$, so that

$$k^2 = \bar{k}^2 = \bar{k}_1^2 + \cdots + \bar{k}_d^2 \quad \text{and} \quad d^dk = id^d\bar{k} \tag{A.8}$$

A.2 Gamma function identities

Expansion near poles:

$$\Gamma(-n+x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma + \sum_{k=1}^{n} k^{-1} + O(x)\right] \tag{A.9}$$

Derivative at positive integers:

$$\Gamma'(m) = (m-1)!\left(-\gamma + \sum_{k=1}^{m-1}\frac{1}{k}\right) \tag{A.10}$$

A.3 Gamma matrix identities

Gamma matrices satisfy the following Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}. \tag{A.11}$$

Some properties of the γ_5:

$$\gamma_5^2 = 1 \tag{A.12}$$

$$\{\gamma^\mu, \gamma_5\} = 0. \tag{A.13}$$

Trace identities:

$$\mathrm{Tr}\,1 = d \tag{A.14}$$

$$\mathrm{Tr}[\text{odd no. of } \gamma^\mu\text{s}] = 0 \tag{A.15}$$

$$\mathrm{Tr}\,\gamma_5 = 0 \tag{A.16}$$

$$\mathrm{Tr}[\gamma_5(\text{odd no. of } \gamma^\mu\text{s})] = 0 \tag{A.17}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = -4g^{\mu\nu} \tag{A.18}$$

$$\mathrm{Tr}[\not{a}\not{b}] = -4(ab) \tag{A.19}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}] \tag{A.20}$$

$$\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4[(ab)(cd) - (ac)(bd) + (ad)(bc)] \tag{A.21}$$

(A.22)

Some contraction identities:

$$\gamma^\mu \gamma_\mu = -d \tag{A.23}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = (d-2)\gamma^\nu \tag{A.24}$$

$$\gamma^\mu \not{a} \gamma_\mu = (d-2)\not{a} \tag{A.25}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho} - (d-4)\gamma^\nu \gamma^\rho \tag{A.26}$$

$$\gamma^\mu \not{a}\not{b} \gamma_\mu = 4(ab) - (d-4)\not{a}\not{b} \tag{A.27}$$

A.4 Group representations

A representation R of a compact nonabelian group is specified by a set of $D(R) \times D(R)$ matrices T^a_R. These matrices satisfy the Lie algebra of the group:

$$[T^a_R, T^b_R] = i f^{abc} T^c_R, \tag{A.28}$$

where the structure coefficients f^{abc} are real and completely antisymmetric.

Adjoint representation of a compact nonabelian group is given by

$$(T^a_A)^{bc} = -i f^{abc}. \tag{A.29}$$

Index of the representation $T(R)$, and the quadratic Casimir $C(R)$, are defined as,

$$\mathrm{Tr}\, T^a_R T^b_R = T(R)\delta^{ab} \quad \text{and} \quad T^a_R T^a_R = C(R), \tag{A.30}$$

respectively. These quantities satisfy $T(R)D(A) = C(R)D(R)$.

For fundamental representation N of special unitary groups SU(N), we have

$$T(N) = \frac{1}{2} \quad \text{and} \quad C(N) = \frac{N^2-1}{2N}, \tag{A.31}$$

and for the adjoint representation $T(A) = N$.

Appendix B

Feynman Rules

Feynman rules for quantum field theories used in the main text have been reproduced here for reference.

B.1 For scalar field theories

Ignoring interactions for a moment, the free scalar propagator in momentum space is given by

$$\Delta(k^2) = \frac{1}{k^2 + M^2 - i\epsilon}, \tag{B.1}$$

and the free fermion propagator in momentum space is given by

$$S(\not{p}) = \frac{-\not{p} + m}{p^2 + m^2 - i\epsilon}. \tag{B.2}$$

In each of the following theories, internal scalar lines carrying momentum k are accompanied by a factor of $-i\Delta(k^2)$, and internal fermion lines carrying momentum p are accompanied by a factor of $-iS(\not{p})$.

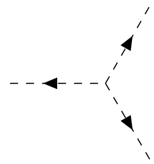
B.1.1 Scalar field theory with a cubic self-interaction

The renormalized Lagrangian that describes this theory is

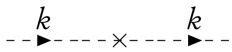
$$L = -\frac{1}{2}Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2}Z_M M^2 \phi^2 + \frac{1}{6}Z_\varkappa \varkappa \phi^3 + Y\phi. \tag{B.3}$$

Due to the cubic self-interaction this theory has just one interaction vertex, at which three scalar lines meet. The Feynman rules are as follows.

1. The expression corresponding to each diagram contains the following pieces:
   - a factor of 1 for each external line,
   - a free field propagator $\Delta(k^2)/i$ for each internal line with momentum k,
   - and the following factors for the vertices



$= i Z_\varkappa \varkappa,$



$= -i(Z_\phi - 1)k^2 - i(Z_M - 1)M^2.$

2. In a tree level calculation, the three point vertex factor should be taken to be $i\varkappa$ and the counterterm vertices should be ignored, as $Z_i = 1 + O(\varkappa^2)$.
3. A diagram with L closed loops will have L undetermined momenta. Each of the undetermined momenta should be integrated over.
4. If there are exchanges of internal propagators and vertices that leave the diagram unchanged, it represents that an overcounting has occurred, and the diagram is said to carry a *symmetry factor*. The final expression should be divided by the symmetry factor.

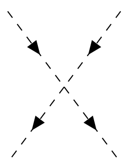
B.1.2 Scalar field theory with a quartic self-interaction

The renormalized Lagrangian for this theory is

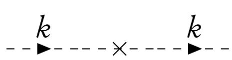
$$L = -\frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_M M^2 \phi^2 - \frac{1}{4!} Z_\lambda \lambda \phi^4. \tag{B.4}$$

Due to the quartic self-interaction this theory has just one interaction vertex at which four scalar lines meet. The Feynman rules are as follows

1. The expression corresponding to each diagram contains the following pieces:
   - a factor of 1 for each external line,
   - a free field propagator $\Delta(k^2)/i$ for each internal line with momentum k,
   - and the following factors for the vertices



$= -i Z_\lambda \lambda,$



$= -i(Z_\phi - 1)k^2 - i(Z_M - 1)M^2.$

2. In a tree level calculation, the four point vertex factor should be taken to be $-i\lambda$ and the counterterm vertices should be ignored, as $Z_i = 1 + O(\lambda)$.
3. A diagram with L closed loops will have L undetermined momenta. Each of the undetermined momenta should be integrated over.
4. If there are exchanges of internal propagators and vertices that leave the diagram unchanged, it represents that an overcounting has occurred, and the diagram is

said to carry a *symmetry factor*. The final expression should be divided by the symmetry factor.

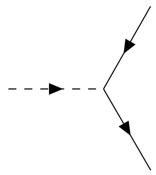
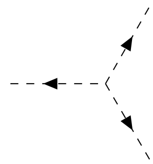
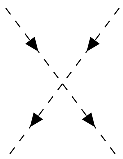
B.1.3 Yukawa theory

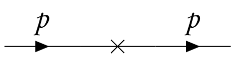
The renormalized Lagrangian for Yukawa theory is

$$L = iZ_\psi \bar{\psi} \partial\!\!\!/ \psi - Z_m m \bar{\psi}\psi - \frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_M M^2 \phi^2 + Z_g g \phi \bar{\psi}\psi + \frac{1}{3!} Z_\varkappa \varkappa \phi^3 - \frac{1}{4!} Z_\lambda \lambda \phi^4 + Y\phi. \tag{B.5}$$

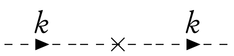
Apart from the Yukawa vertex, self interactions for the scalar field have to be added to make sure that the theory is renormalizable. This leads to three interaction vertices in this theory. The Feynman rules are as follows.

1. The expression corresponding to each diagram contains the following pieces:
   - a factor of 1 for each external scalar,
   - a free field propagator $\Delta(k^2)/i$ for each internal scalar with momentum k,
   - a factor of $u_s(\mathbf{p})$ for each incoming fermion,
   - a factor of $\bar{u}_{s'}(\mathbf{p}')$ for each outgoing fermion,
   - a factor of $\bar{v}_s(\mathbf{p})$ for each incoming antifermion,
   - a factor of $v_{s'}(\mathbf{p}')$ for each outgoing antifermion,
   - a free field propagator $S(p\!\!\!/)/i$ for each internal fermion with momentum p,
   - and the following factors for the vertices

 $= iZ_g g$,  $= iZ_\varkappa \varkappa$,  $= -iZ_\lambda \lambda$,



$= -i(Z_\psi - 1)p\!\!\!/ - (Z_m - 1)m$,



$= -i(Z_\phi - 1)k^2 - i(Z_M - 1)M^2$.

2. In a tree level calculation, the vertex factors should be taken to be ig, $i\varkappa$ and $-i\lambda$ respectively, and the counterterm vertices should be ignored, because $Z_i = 1 + O(g^2, \varkappa^2, \lambda)$.
3. Overall sign of tree diagrams has to be determined by the relative direction of arrows on two fermion lines joined by a scalar.
4. A diagram with L closed loops will have L undetermined momenta. Each of the undetermined momenta should be integrated over.
5. Each closed fermion loop gives a factor of -1.

6. If there are exchanges of internal propagators and vertices that leave the diagram unchanged, it represents that an overcounting has occurred, and the diagram is said to carry a *symmetry factor*. The final expression should be divided by the symmetry factor.

B.2 For electrodynamics

Free photon propagator in R_ξ gauge (in momentum space) is

$$\Delta_{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{B.6}$$

Feynman gauge corresponds to the choice $\xi = 1$ and is convenient for evaluation of loop diagrams in spinor electrodynamics, while the $\xi = 0$ corresponds to the Lorenz gauge, which is convenient for evaluation of loop diagrams in scalar electrodynamics. For completeness, propagators for scalar and spinor fields are given by

$$\Delta(k^2) = \frac{1}{k^2 + M^2 - i\epsilon} \quad \text{and} \quad S(\not{p}) = \frac{-\not{p} + m}{p^2 + m^2 - i\epsilon}, \tag{B.7}$$

respectively.

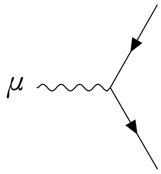
B.2.1 Coupled to spinors

A theory of spinors coupled to the electromagnetic field is described by the following renormalized Lagrangian

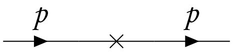
$$L = -\frac{1}{4}Z_3 F^{\mu\nu}F_{\mu\nu} + iZ_2\bar{\psi}\not{\partial}\psi - Z_m m\bar{\psi}\psi + Z_1 e\bar{\psi}\not{A}\psi. \tag{B.8}$$

Apart from the counterterm vertices, this theory has an interaction vertex that connects a photon to a fermion-antifermion pair. The Feynman rules are as follows.

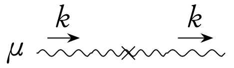
1. The expression corresponding to each diagram contains the following pieces:
   - a factor of $\epsilon_\lambda^{\mu *}(\mathbf{k})$ for each incoming photon,
   - a factor of $\epsilon_{\lambda'}^{\mu}(\mathbf{k}')$ for each outgoing photon,
   - a free field propagator $\Delta_{\mu\nu}(k)/i$ for each internal photon carrying momentum k,
   - a factor of $u_s(\mathbf{p})$ for each incoming fermion,
   - a factor of $\bar{u}_{s'}(\mathbf{p}')$ for each outgoing fermion,
   - a factor of $\bar{v}_s(\mathbf{p})$ for each incoming antifermion,
   - a factor of $v_{s'}(\mathbf{p}')$ for each outgoing antifermion,
   - a free field propagator $S(\not{p})/i$ for each internal fermion with momentum p,
   - and the following factors for the vertices



$$= iZ_1 e\gamma^\mu$$



$$= -i(Z_2 - 1)\not{p} - (Z_m - 1)m,$$



$$= -i(Z_3 - 1)(k^2 g_{\mu\nu} - k_\mu k_\nu).$$

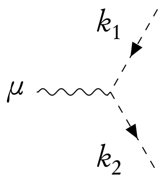
2. In a tree level calculation, the interaction vertex factor should be taken to be $ie\gamma^\mu$ and the counterterms should be ignored, because $Z_i = 1 + O(e^2)$.
3. Overall sign of tree diagrams has to be determined by the relative direction of arrows on two fermion lines joined by a photon.
4. A diagram with L closed loops will have L undetermined momenta. Each of the undetermined momenta should be integrated over.
5. Each closed fermion loop gives a factor of -1.
6. If there are exchanges of internal propagators and vertices that leave the diagram unchanged, it represents that an overcounting has occurred, and the diagram is said to carry a *symmetry factor*. The final expression should be divided by the symmetry factor.

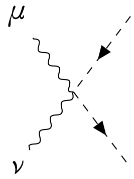
B.2.2 Coupled to scalars

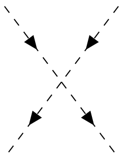
A theory of complex scalars coupled to the electromagnetic field is described by the following renormalized Lagrangian

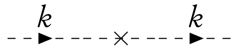
$$\begin{aligned} L = &-\frac{1}{4}Z_3 F^{\mu\nu}F_{\mu\nu} - Z_2 \partial^\mu \phi^\dagger \partial_\mu \phi - Z_M M^2 \phi^\dagger \phi \\ &-\frac{1}{4}Z_\lambda \lambda(\phi^\dagger\phi)^2 + iZ_1 e A^\mu\left[\phi^\dagger(\partial_\mu\phi) - (\partial_\mu\phi^\dagger)\phi\right] - Z_4 e^2 \phi^\dagger\phi A^\mu A_\mu. \end{aligned} \tag{B.9}$$

1. The expression corresponding to each diagram contains the following pieces:
   - a factor of $\epsilon_\lambda^{\mu*}(\mathbf{k})$ for each incoming photon,
   - a factor of $\epsilon_{\lambda'}^{\mu}(\mathbf{k}')$ for each outgoing photon,
   - a free field propagator $\Delta_{\mu\nu}(k)/i$ for each internal photon carrying momentum k,
   - a factor of 1 for each external scalar line
   - a free field propagator $\Delta(k^2)/i$ for each internal fermion with momentum k,
   - and the following factors for the vertices

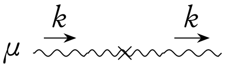


$= iZ_1 e(k_1+k_2)_\mu,$


$= -2iZ_4 e^2 g_{\mu\nu},$

$= -iZ_\lambda \lambda,$



$= -i(Z_2-1)k^2 - i(Z_M-1)M^2,$



$= -i(Z_3-1)(k^2 g_{\mu\nu} - k_\mu k_\nu).$

2. In a tree level calculation, the interaction vertex factors should be taken to be $ie(k_1+k_2)_\mu$, $-2i^2 g_{\mu\nu}$, and $-i\lambda$ respectively, and counterterms should be ignored, because $Z_i = 1 + O(e^2, \lambda)$.
3. A diagram with L closed loops will have L undetermined momenta. Each of the undetermined momenta should be integrated over.
4. If there are exchanges of internal propagators and vertices that leave the diagram unchanged, it represents that an overcounting has occurred, and the diagram is said to carry a *symmetry factor*. The final expression should be divided by the symmetry factor.

B.3 For nonabelian gauge theory

The Yang-Mills Lagrangian along with a gauge fixing term give

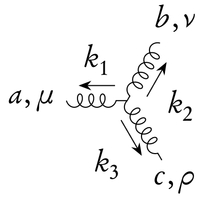
$$\begin{aligned} L_{\text{YM}} + L_{\text{gf}} =& \frac{1}{2} A^{e\mu}(g_{\mu\nu} - \partial_\mu \partial_\nu) A^{e\nu} + \frac{1}{2}\xi^{-1} A^{e\mu}\partial_\mu \partial_\nu A^{e\nu} \\ & - g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^c_\nu - \frac{1}{4} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu . \end{aligned} \tag{B.10}$$

If we ignore gluon self-interactions for a moment, the first line gives the free gluon propagator in the R_ξ gauge,

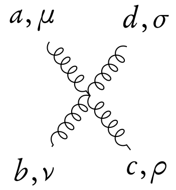
$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2 - i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2} \right). \tag{B.11}$$

Choice of $\xi = 1$ corresponds to the Feynman gauge and proves to be convenient for loop calculations when the gauge field is coupled to spinors, and $\xi = 0$ corresponds to the Lorenz gauge and proves to be convenient for loop calculations when the gauge field is coupled to scalars.

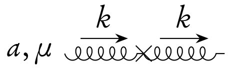
Vertices for gluon self-interactions are the same, whether the theory is coupled to spinors or scalars. Vertex factors are as follows.



$$= i\mathbf{V}^{abc}_{\mu\nu\rho}(k_1,k_2,k_3)$$
$$= gf^{abc}[(k_1-k_2)_\rho g_{\mu\nu} + (k_2-k_3)_\mu g_{\nu\rho} + (k_3-k_1)_\nu g_{\rho\mu}],$$



$$= i\mathbf{V}^{abcd}_{\mu\nu\rho\sigma} = -ig^2\big[f^{abe}f^{cde}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) + f^{ace}f^{dbe}(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma}) + f^{ade}f^{bce}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu})\big],$$



$$= -i(Z_3-1)(k^2 g_{\mu\nu} - k_\mu k_\nu)\delta^{ab}.$$

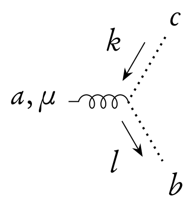
Gauge fixing also leads to the introduction of ghosts. The ghost Lagrangian is

$$L_{\text{gh}} = -\partial^\mu \bar{c}^c \partial_\mu c^c + gf^{abc}A^a_\mu \partial^\mu \bar{c}^b c^c. \tag{B.12}$$

The kinetic part of the Lagrangian gives the ghost propagator

$$\Delta^{ab}(k^2) = \frac{\delta^{ab}}{k^2 - i\epsilon}. \tag{B.13}$$

Since ghosts are Grassmann fields, diagrams with a closed ghost loop will receive a factor of -1. Interaction of ghosts with gauge fields leads the the following vertex



$$= gf^{abc}l_\mu$$

B.3.1 Coupled to spinors

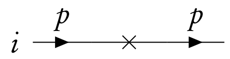
Spinors in representation R of the gauge group carry a colour index, and couple to the gauge field via the following Lagrangian

$$L_{\text{fermion}} = iZ_2\bar{\psi}^i \partial\!\!\!/ \psi_i - Z_m m\bar{\psi}^i\psi_i + Z_1 g A^{a\mu}\bar{\psi}^i\gamma^\mu (T^a_R)_i^{\ j}\psi_j, \tag{B.14}$$

where T^a_R are generators of the gauge group in representation R. Number of colours is equal to the dimension of the representation. Due to colours, the fermion propagator carries additional indices,

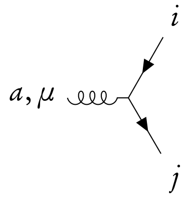
$$S_i^{\ j}(\not{p}) = \frac{(-\not{p} + m)\delta_i^{\ j}}{p^2 + m^2 - i\epsilon}, \tag{B.15}$$

and so does the counterterm for fermion propagator,



$$= (-i(Z_2 - 1)\not{p} - i(Z_m - 1)m)\delta_i^{\ j}.$$

Rules for incoming and outgoing fermions is identical to electrodynamics. The interaction vertex gives the following factor



$$= i(\mathbf{V}^{a\mu})_i^{\ j} = iZ_1 g\gamma^\mu (T_R^a)_i^{\ j}$$

B.3.2 Coupled to scalars

Complex scalars in representation R of the gauge group carry a colour index, and couple to the gauge field via the following Lagrangian

$$\begin{aligned} L_{\text{scalar}} = & -Z_2 \partial^\mu \phi^{\dagger i} \partial_\mu \phi_i - Z_M M^2 \phi^{\dagger i} \phi_i \\ & -\frac{1}{2} Z_\lambda \lambda \phi^{\dagger i} \phi_i \phi^{\dagger j} \phi_j + igZ_1 A_\mu^a \Big[(\partial^\mu \phi^{\dagger i})(T_R^a)_i^{\ j} \phi_j - \phi^{\dagger i} (T_R^a)_i^{\ j} (\partial^\mu \phi_j) \Big] \\ & -Z_4 g^2 A^{a\mu} A_\mu^b \phi^{\dagger i} (T_R^a T_R^b)_i^{\ j} \phi_j, \end{aligned} \tag{B.16}$$

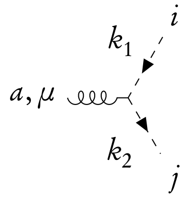
where T_R^a are generators of the gauge group in representation R. Number of colours is equal to the dimension of the representation. Due to colours, the scalar propagator carries additional indices,

$$\Delta_i^{\ j}(k^2) = \frac{\delta_i^{\ j}}{k^2 + M^2 - i\epsilon}, \tag{B.17}$$

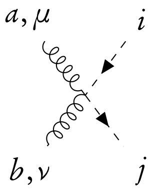
and so does the counterterm for scalar propagator,

$$= (-i(Z_2 - 1)k^2 - i(Z_M - 1)M^2)\delta_i^{\ j}.$$

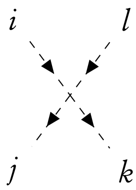
Rules for incoming and outgoing scalars is identical to electrodynamics. Interactions give the following vertex factors



$$= iZ_1 g(k_1 + k_2)_\mu (T_R^a)_i^{\ j},$$



$$= -iZ_4 g^2 (T_R^a T_R^a + T_R^b T_R^a)_i^{\ j} g_{\mu\nu},$$



$$= -\frac{i}{2} Z_\lambda \lambda (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}).$$

Appendix C

Coloured Ordered Feynman Rules

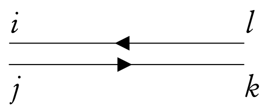
The Yang–Mills Lagrangian in the Gervais–Neveu gauge is of the form,

$$L = \mathrm{Tr}\Big(-\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - i\sqrt{2}g\,\partial^\mu A^\nu A_\nu A_\mu + \frac{1}{4}g^2 A^\mu A^\nu A_\mu A_\nu\Big). \tag{C.1}$$

Treating A^μ as a matrix field, $(A^\mu)_i^{\ j} = A^{a\mu}(T^a)_i^{\ j}$, the propagator for $(A^\mu)_i^{\ j}$ is

$$(\Delta^{ab}_{\mu\nu})_{i\ j}^{\ k\ l}(k^2) = (T^a)_i^{\ j}(T^b)_k^{\ l}\,\frac{\delta^{ab}g_{\mu\nu}}{k^2 - i\epsilon}. \tag{C.2}$$

In the double line notation, the gluon propagator is,

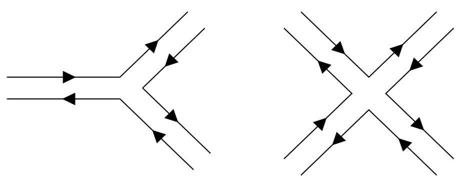


(C.3)

Arrows point from an up index to a down index. Having taken colour factors into account already, the vertex rules become very simple

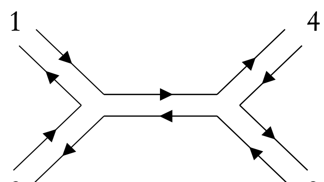
- 3-point vertex $\mathbf{V}_{\mu\nu\rho}(p,q,r) = -i\sqrt{2}g(g_{\mu\nu}p_\rho + g_{\nu\rho}q_\mu + g_{\rho\mu}r_\nu)$,
- 4-point vertex $\mathbf{V}_{\mu\nu\rho\sigma} = i g^2 g_{\mu\rho} g_{\nu\sigma}$.

Due to the way colour indices are contracted the vertices in double line notation look like



(C.4)

Colour factors are assigned by starting at an external point and following arrows backwards. For example, in the following diagram



(C.5)

$\approx 100\zeta(3)$

the colour factor is $\mathrm{Tr}\, T^{a_1} T^{a_2} T^{a_3} T^{a_4}$. Similarly, in general for a diagram with n external gluons, the colour factor will be $\mathrm{Tr}\, T^{a_1} T^{a_2} \cdots T^{a_n}$.

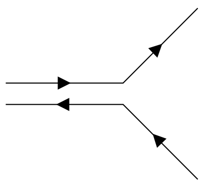
As a final note, when there are massless quarks in the theory that couple to gauge fields by an interaction term of the form $L_1 = i(g/\sqrt{2})\bar{\psi}\not{A}\psi$, the fermion propagator will just be a single line, but will carry colour indices,

$$S_i^{\ j}(\not{p}) = \frac{-\not{p}}{p^2 - i\epsilon}\delta_i^{\ j}. \tag{C.6}$$

The vertex factor will be,

– Quark-gluon vertex, $i\mathbf{V}^\mu = \frac{ig}{\sqrt{2}}\gamma^\mu$,

and finally in double line notation, the quark gluon vertex will look like



(C.7)

where the single lines are for quarks and arrows are drawn so that they are consistent with arrows on fermion lines in traditional Feynman diagrams.