Universal Coefficient Theorem (for Homology)

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March 8, 2024

Let X be a topological space and G be an abelian group; and as in usual singular homology, let $\sigma^n : \Delta^n \to X$ be the singular n-simplices. Define groups $C_n(X;G)$ as

$$C_n(X;G) \coloneqq \left\{ \sum_{i=1}^{N} g_i \sigma_i^n : N \in \mathbb{Z}_+,\ \sigma_i^n \text{ are } n\text{-simplices},\ g_i \in G \right\}, \tag{1}$$

and the operator ∂_n^G by its action on simplices, $\partial_n^G \left(\sum_i g_i \sigma_i^n\right) \coloneqq \sum_i g_i \left(\partial_n \sigma_i^n\right)$.

Exercise 1. $\cdots \xrightarrow{\partial_{n+2}^G} C_{n+1}(X;G) \xrightarrow{\partial_{n+1}^G} C_n(X;G) \xrightarrow{\partial_n^G} C_{n-1}(X;G) \xrightarrow{\partial_{n-1}^G} \cdots$ *is a chain complex, i.e., show that ∂_n^G are group homomorphisms, and $\partial_n^G \circ \partial_{n-1}^G = 0$.*

Denote homology groups of this complex as $H_n(X;G)$, and call them *singular homology groups of X with coefficients in G*.

Exercise 2. *If X is a point, then $H_n(X;G) = 0$ for $n > 0$, and $H_0(X;G) = G$.*

Exercise 3. *Suppose G_1, G_2 are abelian groups, and $\phi : G_1 \to G_2$ is a group homomorphism. Show that induced homomorphism $\phi_\sharp : C_n(X;G_1) \to C_n(X;G_2)$ is a chain map, and therefore descends to a morphism of homology groups $\phi_* : H_n(X;G_1) \to H_n(X;G_2)$.*

There is a natural isomorphism $C_n(G) \xrightarrow{\sim} C_n \otimes G$ given by $\sum_i g_i \sigma_i^n \mapsto \sum_i \sigma_i^n \otimes g_i$ and extended by $\mathbb{Z}$-linearity. Note that, with respect to this tensor product, the boundary map defined above decomposes as $\partial^G = \partial \otimes \mathrm{id}_G$.

Now, the problem we would like to address is the following: Given a chain complex $\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$, and an abelian group G, we would like to compute the homology of chain complex,

$$\cdots \xrightarrow{\partial \otimes \mathrm{id}_G} C_{n+1} \otimes G \xrightarrow{\partial \otimes \mathrm{id}_G} C_n \otimes G \xrightarrow{\partial \otimes \mathrm{id}_G} C_{n-1} \otimes G \xrightarrow{\partial \otimes \mathrm{id}_G} \cdots,$$

in terms of $H_n(C)$ and G.

At this point it is natural to ask: why isn't $H_n(C;G) = H_n(C) \otimes G$? A succinct, but rather unilluminating, answer would be: because the functor $\otimes G$ is *not exact*. Here is a motivating example that illustrates this.

Suppose, we have the following short exact sequence of abelian groups, $0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$. If it were to be tensored with $\mathbb{Z}/2\mathbb{Z}$, we would have the following sequence, $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$, which fails to be exact at the first $\mathbb{Z}/2\mathbb{Z}$.

Homology groups $H_n(C;G)$ fit into the following exact sequences,

$$0 \longrightarrow B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}_G} Z_n \otimes G \longrightarrow H_n(C;G) \longrightarrow 0,$$

however, if I start with the short exact sequence $0 \to B_n \to Z_n \to H_n(C) \to 0$, and tensor with G, the resulting sequence,

$$0 \longrightarrow B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}_G} Z_n \otimes G \longrightarrow H_n(C) \otimes G \longrightarrow 0, \tag{2}$$

need not be exact, as seen in the $\mathbb{Z}/2\mathbb{Z}$ example above. As we shall see later, even though tensor product is not exact, it is *right exact*, i.e., if the sequence of abelian groups $A \to B \to C \to 0$ is exact, then so is $A \otimes G \to B \otimes G \to C \otimes G \to 0$.

The failure of (2) to be exact—and therefore the reason for $H_n(C;G) \neq H_n(C) \otimes G$—lies at $\to B_n \otimes G \to$, and is measured by the group $\ker(i_n \otimes \mathrm{id}_G)$ [1]. Indeed, (2) becomes exact at the insertion of this extra term,

$$0 \longrightarrow \ker(i_n \otimes \mathrm{id}_G) \longrightarrow B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}_G} Z_n \otimes G \longrightarrow H_n(C) \otimes G \longrightarrow 0.$$

As before, let $\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots$ be a chain complex, and define *cycles*, $Z_n = \ker \partial_n$, and *boundaries*, $B_n = \mathrm{im}\, \partial_{n+1}$. Note that $B_n \subset Z_n \subset C_n$.

Exercise 4. *Verify the following:*

1. $\cdots \xrightarrow{\partial} Z_n \xrightarrow{\partial} \cdots$ *and* $\cdots \xrightarrow{\partial} B_n \xrightarrow{\partial} \cdots$ *are chain complexes and* $0 \to Z_n \hookrightarrow C_n \xrightarrow{\partial} B_{n-1} \to 0$ *is a split short exact sequence of chain complexes.*
2. $\cdots \xrightarrow{\partial \otimes \mathrm{id}_G} Z_n \otimes G \xrightarrow{\partial \otimes \mathrm{id}_G} \cdots$ *is a chain complex, and* $H_n(Z \otimes G) = Z_n \otimes G$.
3. $\cdots \xrightarrow{\partial \otimes \mathrm{id}_G} B_n \otimes G \xrightarrow{\partial \otimes \mathrm{id}_G} \cdots$ *is a chain complex, and* $H_n(B \otimes G) = B_n \otimes G$.

Tensoring the short exact sequence in (1) above with G gives,

$$0 \to Z_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}_G} C_n \otimes G \xrightarrow{\partial \otimes \mathrm{id}_G} B_{n-1} \otimes G \to 0, \tag{3}$$

which, in general, may not be exact. However, since the original sequence splits, $C_n = Z_n \oplus B_{n-1}$, and we have the natural isomorphism, $C_n \otimes G = Z_n \otimes G \oplus B_{n-1} \otimes G$, the new sequence (3) is split exact as well.

This short exact sequence of complexes induces a long exact sequence in homology,

$$\cdots \longrightarrow B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}_G} Z_n \otimes G \longrightarrow H_n(C;G) \longrightarrow B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathrm{id}_G} Z_{n-1} \otimes G \longrightarrow \cdots,$$

[1] Often denoted $\mathrm{Tor}(H_n(C), G)$

where $i_n : B_n \to Z_n$ is the usual inclusion, and the boundary maps $i_n \otimes \mathrm{id}_G$ are computed from an application of the *snake lemma*.

Long exact sequence above can be broken into the following short exact sequence,

$$0 \longrightarrow \mathrm{coker}(i_n \otimes \mathrm{id}_G) \longrightarrow H_n(C;G) \longrightarrow \ker(i_{n-1} \otimes \mathrm{id}_G) \longrightarrow 0,$$

where $\mathrm{coker}(i_n \otimes \mathrm{id}_G) = Z_n \otimes G/\mathrm{im}(i_n \otimes id_G)$.

Lemma 1. *If the sequence of abelian groups* $A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ *is exact, then so is* $A \otimes G \xrightarrow{i \otimes \mathrm{id}_G} B \otimes G \xrightarrow{j \otimes \mathrm{id}_G} C \otimes G \longrightarrow 0$.

In particular, the above lemma implies that there is a natural isomorphism $C \otimes G = B \otimes G/\mathrm{im}(i \otimes \mathrm{id}_G)$; so that with $A = B_n$, $B = Z_n$, and $C = H_n(C)$, we have $\mathrm{coker}(i_n \otimes \mathrm{id}_G) = H_n(C) \otimes G$.

If we recall the notation for $\ker(i_n \otimes \mathrm{id}_G)$ introduced in the previous section, we have the following short exact sequence,

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C;G) \longrightarrow \mathrm{Tor}(H_{n-1}(C), G) \longrightarrow 0, \tag{4}$$

which splits. This is the content of *universal coefficient theorem* (homology version).

To see that (4) splits, recall that the sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$ splits so that there is a projection map $p : C_n \to Z_n$ that restricts to the identity on Z_n ($p \circ i = \mathrm{id}_{Z_n}$). Composing with the surjection $Z_n \to H_n(C)$, we have a map $C_n \to H_n(C)$. If we think of $\xrightarrow{\partial} H_n(C) \xrightarrow{\partial}$ as a (very silly) chain complex, $C \to H(C)$ can be seen to satisfy the chain map condition. Tensoring with G to get another chain map $C \otimes G \to H(C) \otimes G$, and looking at its descendent in homology $H_n(C;G) \to H_n(C) \otimes G$ gives the required splitting.

Something about free resolution of abelian groups, Tor *as homology of a chain complex, independence from the choice of free resolution, properties of* Tor*, and computations (see Hatcher).*
