Notes on the Schrödinger-Lichnerowicz-Weitzenböck Formula

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Given a vector bundle $E \to M$ with a connection $\nabla^E : \Gamma(E) \to \Gamma(T^*M \otimes E)$, one can always define a *connection Laplacian*, $\nabla^{E*}\nabla^E : \Gamma(E) \to \Gamma(E)$. In a local frame $\{e_\mu\}$ for the tangent bundle,

$$\nabla^{E*}\nabla^E = -\sum_\mu \nabla^E_\mu \nabla^E_\mu + \sum_\mu \nabla^E_{\nabla_\mu e_\mu} \tag{1}$$

$$= -\sum_\mu \nabla^E_\mu \nabla^E_\mu - \sum_{\mu,\nu} g(\nabla_\nu e_\mu, e_\nu)\nabla^E_\mu. \tag{2}$$

When $E = \bigwedge T^*M$, there is another notion of Laplacian: the *Hodge Laplacian* or the *Laplace-de Rham operator*, $\Delta = d^*d + dd^*$.

It is reasonable to ask how these two second order operators, both called Laplacians, are related. The answer is given by the *Weitzenböck identity* (Weitzenböck 1923)

$$\Delta = \nabla^*\nabla + \mathscr{R}, \tag{3}$$

where $\mathscr{R}$ is a zero order operator that depends only on the curvature of M.

In spin geometry, one expects square of the Dirac operator to be a Laplacian, and one may ask how it is related to the spin connection Laplacian. The answer here is given by the *Schrödinger-Lichnerowicz-Weitzenböck formula* (Schrödinger 1932, Lichnerowicz 1963),

$$\not{D}^2 = \nabla^*\nabla + \frac{1}{4}R, \tag{4}$$

where ∇ is induced on the spinor bundle by the Levi-Civita connection, R is the scalar curvature of M.

This already has the following consequences:

Corollary. *Any compact spin manifold of positive scalar curvature admits no nonzero harmonic spinors.*

Corollary. *On a compact spin manifold with $R \equiv 0$, harmonic spinors are globally parallel, i.e., $\psi \in \ker \not{D} \implies \nabla\psi \equiv 0$.*

When spinors are coupled to a gauge field A, there is the following generalization,

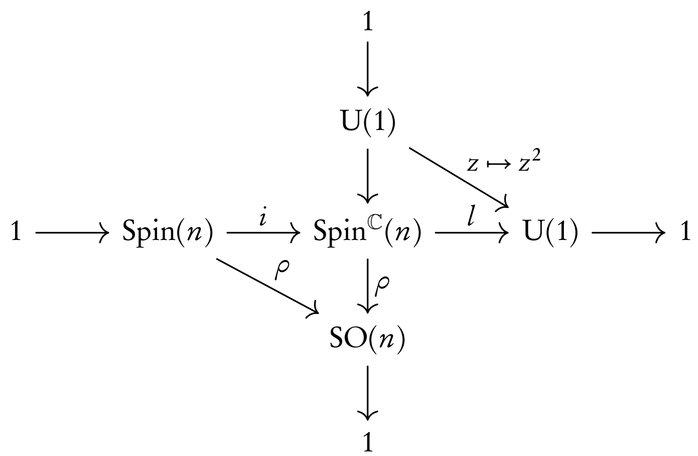
$$D\!\!\!/_A^2 = \nabla^{A*}\nabla^A + \frac{1}{4}R + \frac{1}{2}F, \tag{5}$$

where $F = dA$ is the *gauge field strength.*

1 Notation

There are Lie group homomorphisms

- $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$,
- $\rho : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n)$ given by $\rho[g,z] = \rho(g)$,
- $l : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{U}(1)$ given by $l[g,z] = z^2$,
- $p : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n) \times \mathrm{U}(1)$ given by $p[g,z] = (\rho(g), z^2)$.

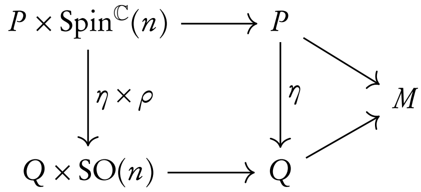


Corresponding Lie algebra homomorphisms can be obtained by differentiation. Recall $\mathfrak{spin}(n) = \mathrm{span}\{e_\mu e_\nu \in \mathscr{C}_n\ : 1 \le \mu < \nu \le n\}$ and $\mathfrak{spin}^{\mathbb{C}}(n) = \mathfrak{spin}(n) \oplus i\mathbb{R}$.

- $\rho_* : \mathfrak{spin}(n) \to \mathfrak{so}(n)$ given by $\rho_* e_\mu e_\nu = 2E_{\mu\nu}$,
- $\rho_* : \mathfrak{spin}^{\mathbb{C}}(n) \to \mathfrak{so}(n)$ given by $\rho_*(e_\mu, e_\nu, it) = 2E_{\mu\nu}$,
- $p_* : \mathfrak{spin}^{\mathbb{C}}(n) \to \mathfrak{so}(n) \oplus i\mathbb{R}$ given by $\rho_*(e_\mu e_\nu, it) = (2E_{\mu\nu}, 2it)$.

Suppose (M, g) is an orientable Riemannian n-manifold, $Q \to M$ the principal $\mathrm{SO}(n)$-bundle of oriented orthonormal frames.

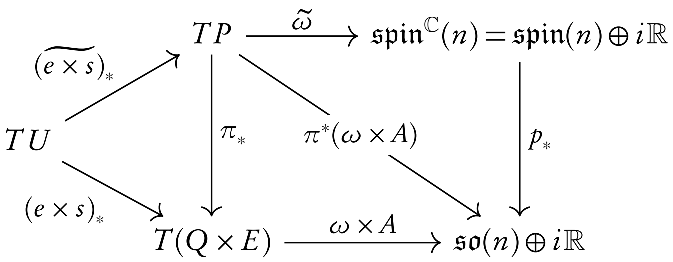
Let (P, η) be a spin$^{\mathbb{C}}$ structure on $Q \to M$. Then there is a fibre-preserving action of $\mathrm{U}(1) \subset \mathrm{Spin}^{\mathbb{C}}(n)$ on P, and $P/\mathrm{U}(1) \cong Q$. Similarly, quotienting P by the $\mathrm{Spin}(n) \subset \mathrm{Spin}^{\mathbb{C}}(n)$ action gives a principal $\mathrm{U}(1)$-bundle $E := P/\mathrm{Spin}(n) \to M$. The fibre-product $Q \times E \to M$ is a principal $\mathrm{SO}(n) \times \mathrm{U}(1)$-bundle and there is a two-fold covering bundle morphism $\pi : P \to Q \times E$ induced by $p : \mathrm{Spin}^{\mathbb{C}} \to \mathrm{SO}(n) \times \mathrm{U}(1)$.



The complex Clifford algebra $\mathscr{C}_n^{\mathbb{C}}$ comes with a Dirac representation $\gamma : \mathscr{C}_n^{\mathbb{C}} \to \mathrm{End}(\Delta_n)$, where $\Delta_n = \mathbb{C}^{2^k}$ for $n = 2k, 2k+1$. γ restricts to representations of the real Clifford algebra, Pin, Spin groups and Lie algebras. The restriction $\gamma : \mathrm{Spin}(n) \to \mathrm{Aut}(\Delta_n)$ is used to construct the *spinor bundle* $\Sigma := P \times_\gamma \Delta_n$.

2 Connection

Suppose $\omega : TQ \to \mathfrak{so}(n)$ is the Levi-Civita connection on $Q \to M$, and also fix a connection $A : TE \to i\mathbb{R}$ on E (where we have identified $\mathfrak{u}(1)$ with $i\mathbb{R}$). Denote by $\omega \times A : T(Q \times E) \to \mathfrak{so}(n) \oplus i\mathbb{R}$, the connection induced on $Q \times E \to M$. $\omega \times A$ can be pulled back to $\pi^*(\omega \times A) : TP \to \mathfrak{so}(n) \oplus i\mathbb{R}$, and composed with p_*^{-1} to get a connection $\widetilde{\omega} := p_*^{-1}\pi^*(\omega \times A) : TP \to \mathfrak{spin}^{\mathbb{C}}(n)$ on P.



If in local frames $e : U \to Q$ and $s : U \to E$,

$$e^*\omega = \sum_{\mu<\nu} \omega^e_{\mu\nu} E_{\mu\nu} \quad \text{and} \quad s^*A = A^s, \tag{6}$$

then in the local frame $e \times s : U \to Q \times E$,

$$(e \times s)^*(\omega \times A) = \left(\sum_{\mu<\nu} \omega^e_{\mu\nu} E_{\mu\nu}, A^s \right). \tag{7}$$

Moreover, if $\widetilde{e \times s} : U \to P$ is a lift of $e \times s$ such that $e \times s = \pi \circ \widetilde{e \times s}$, then

$$(e \times s)^*(\omega \times A) = p_*(\widetilde{e \times s})^*\widetilde{\omega} = \left(\sum_{\mu<\nu} \omega^e_{\mu\nu} E_{\mu\nu}, A^s \right), \tag{8}$$

and therefore

$$(\widetilde{e \times s})^*\widetilde{\omega} = \left(\frac{1}{2} \sum_{\mu<\nu} \omega^e_{\mu\nu} e_\mu e_\nu, \frac{1}{2} A^s \right). \tag{9}$$

For a spinor $\psi : U \to \Sigma$, the covariant derivative with respect to this connection is

$$\nabla^A \psi = d\psi + \frac{1}{2}\sum_{\mu<\nu} \omega^e_{\mu\nu} e_\mu e_\nu \psi + \frac{1}{2} A^s \psi. \tag{10}$$

3 Operators

The spin connection Laplacian (in local coordinates)

$$\Delta_A \psi = \nabla^{A*}\nabla^A = -\sum_\mu \nabla^A_\mu \nabla^A_\mu - \sum_{\mu,\nu} g(\nabla_\nu e_\mu, e_\nu)\nabla^A_\mu. \tag{11}$$

Dirac operator on $\Gamma(\Sigma)$ is constructed by composing Clifford multiplication with the spin connection. In local coordinates

$$\not{D}_A \psi = \sum_\mu e_\mu \cdot \nabla^A_\mu \psi. \tag{12}$$

4 Curvature

$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ and $F = dA$ are curvature 2-forms of ω and A respectively. If $\widetilde{\Omega} = d\widetilde{\omega} + \frac{1}{2}[\widetilde{\omega}, \widetilde{\omega}]$, then it can be easily checked that

$$p_* \widetilde{\Omega} = (\pi^*\Omega, \pi^* F). \tag{13}$$

In local coordinates, let

$$e^*\Omega = \sum_{\mu<\nu} \Omega^e_{\mu\nu} E_{\mu\nu}, \qquad s^* F = s^* dA = dA^s, \tag{14}$$

then

$$(\widetilde{e \times s})^* \widetilde{\Omega} = \left(\frac{1}{2}\sum_{\mu<\nu} \pi^* \Omega^e_{\mu\nu} e_\mu e_\nu, \frac{1}{2}\pi^* F^s \right), \tag{15}$$

and

$$\nabla^A(\nabla^A \psi) = (\widetilde{e \times s})^* \widetilde{\Omega} \psi = \frac{1}{2}\sum_{\mu<\nu} \Omega^e_{\mu\nu} e_\mu e_\nu \psi + \frac{1}{2} F^s \psi. \tag{16}$$

Let ∇ be the Levi-Civita connection on $TM \to M$ induced by $\omega : TQ \to \mathfrak{so}(n)$. Then the Riemann curvature operator is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{17}$$

and the Riemann curvature tensor as

$$\mathrm{Rm}(X,Y,Z,W) = g(R(X,Y)Z, W). \tag{18}$$

In local coordinates $R_{\mu\nu\lambda\rho} := \mathrm{Rm}(e_\mu, e_\nu, e_\lambda, e_\rho)$. Also take this opportunity to recall that the Ricci tensor is defined as

$$\mathrm{Ric}(X,Y)=\sum_{\mu}\mathrm{Rm}(e_\mu,X,Y,e_\mu), \tag{19}$$

or in local coordinates $R_{\mu\nu}=\sum_\lambda R_{\lambda\mu\nu\lambda}$. It can also be viewed as an endomorphism $\mathrm{Ric}: TM\to TM$ as $\mathrm{Ric}(X)=\mathrm{Ric}(X,\cdot)^\sharp$. Finally, there is the scalar curvature, $R=\sum_\nu R_{\nu\nu}=\sum_{\mu,\nu}R_{\mu\nu\nu\mu}$.

Recall, symmetries of the curvature tensor

- $R_{\mu\nu\lambda\rho}=-R_{\nu\mu\lambda\rho}$
- $R_{\mu\nu\lambda\rho}=-R_{\mu\nu\rho\lambda}$
- $R_{\mu\nu\lambda\rho}+R_{\nu\lambda\mu\rho}+R_{\lambda\mu\nu\rho}=0$
- $R_{\mu\nu\lambda\rho}=R_{\lambda\rho\mu\nu}$

By the definition of the curvature 2-form, we have $R(X,Y)e_\lambda=\sum_\rho e^*\Omega_{\lambda\rho}(X,Y)e_\rho$, and therefore

$$e^*\Omega_{\mu\nu}(X,Y)=g(R(X,Y)e_\mu,e_\nu)=\frac{1}{2}\sum_{\lambda,\rho}R_{\mu\nu\lambda\rho}e^\lambda\wedge e^\rho(X,Y), \tag{20}$$

where $\{e^\mu\}$ is the frame dual to $\{e_\mu\}$. This helps us write an explicit expression for $\nabla^A\nabla^A\psi$ in local coordinates

$$\nabla^A\nabla^A\psi=\frac{1}{4}\sum_{\mu<\nu}\left(\sum_{\lambda,\rho}R_{\mu\nu\lambda\rho}e^\lambda\wedge e^\rho\right)e_\mu e_\nu\psi+\frac{1}{2}F^s\psi \tag{21}$$

In analogy with the Riemann curvature for the tangent bundle, we can define $R^\Sigma(X,Y)\psi:=\nabla^A\nabla^A\psi(X,Y)=\nabla^A_X\nabla^A_Y\psi-\nabla^A_Y\nabla^A_X\psi-\nabla^A_{[X,Y]}\psi$ for the spinor bundle $\Sigma\to M$. In coordinates

$$R^\Sigma(X,Y)\psi=\frac{1}{4}\sum_{\mu<\nu}\left(\sum_{\lambda,\rho}R_{\mu\nu\lambda\rho}e^\lambda\wedge e^\rho(X,Y)\right)e_\mu e_\nu\psi+\frac{1}{2}F^s(X,Y)\psi. \tag{22}$$

Lemma. *With notation established the prequel,*

$$\sum_\mu e_\mu\cdot R^\Sigma(X,e_\mu)\psi=-\frac{1}{2}\mathrm{Ric}(X)\cdot\psi+\frac{1}{2}i(X)F^s\cdot\psi. \tag{23}$$

Proof is by a computation in local coordinates, but it is not hard to believe that this is true. The first term is a kind of trace over Rm so appearance of the Ricci tensor is not a surprise; the second term requires a short calculation.

5 Formula

Theorem. *Suppose that* (M, g) *is a Riemannian manifold with a spin*$^{\mathbb{C}}$ *structure. Denote by* R *the scalar curvature of* M*, and let* $F = dA$ *be the curvature 2-form of the connection* A *in the* $\mathrm{U}(1)$ *bundle associated with the spin*$^{\mathbb{C}}$ *structure. Then one has*

$$\not{D}_A^2 \psi = \Delta_A \psi + \frac{1}{4} R \psi + \frac{1}{2} F \cdot \psi. \tag{24}$$

Proof. Computation in local coordinates. □