Orthonormal Bases of Hilbert Spaces, Tensor Products, and Representations of $\mathfrak{sl}(2, \mathbb{C})$

Ayush Singh

June 11, 2021

In this project report, we start by reviewing a few topological notions for an inner product space, and then move on to a discussion of orthonormal systems and orthonormal bases in a Hilbert space. We encounter Bessel's inequality, Parseval's identity, and see a proof of existence of an orthonormal basis for an arbitrary Hilbert space. Tensor product of Hilbert spaces is also discussed, and a few of its properties obtained. In particular, we derive a basis of the tensor product space and show that it is spanned by simple tensors.

In the second part, representations of $\mathfrak{sl}(2, \mathbb{C})$ are discussed. After reviewing a few properties of Lie algebras, all finite dimensional irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ are characterised. We end with a result which gives a prescription for decomposing a tensor product of representations of $\mathfrak{sl}(2, \mathbb{C})$ into a direct sum of representations.

1 Hilbert Spaces

To state results for both real and complex vector spaces simultaneously, we shall use the symbol $\mathbb{F}$ for the underlying field. $\mathbb{F}$ could be understood to be $\mathbb{R}$ or $\mathbb{C}$ as appropriate.

1.1 Inner Products

Definition 1 (Sesquilinear form). *Let H be a vector space over $\mathbb{F}$. A mapping $s : H \times H \to \mathbb{F}$ is called a* sesquilinear form *on H if for every $u, v, w \in H$ and $a, b \in \mathbb{F}$ we have*

$$s(au + bv, w) = as(u, w) + bs(v, w), \tag{1}$$

$$s(u, av + bw) = a^* s(u, v) + b^* s(u, w). \tag{2}$$

In case of a real vector space, s as defined above is a bilinear form.

A sesquilinear form s on H is said to be *Hermitian* if for every $u, v \in H$ we have $s(u, v) = s(v, u)^*$. A Hermitian bilinear form on a real vector space is said to be *symmetric* as $s(u, v) = s(v, u)$.

If s is a sesquilinear for on H, then the mapping $q : H \to \mathbb{F}$ defined by $q(u) = s(u, u)$

for each $u \in H$ is called the *quadratic form* on H induced by s. Each quadratic form q satisfies

$$q(au) = s(au, au) = aa^* s(u, u) = |a|^2 q(u) \qquad \text{for every } u \in H, a \in \mathbb{F}. \tag{3}$$

If s is a Hermitian sesquilinear form, and q is the quadratic form induced by s, then it is seen that $q(u) \in \mathbb{R}$ for every $u \in H$; q is *real*.

A Hermitian sesquilinear for is said to be *non-negative* when the quadratic form induced by it is non-negative, i.e.,

$$q(u) = s(u, u) \geq 0 \qquad \text{for every } u \in H; \tag{4}$$

it is said to be *positive* when

$$q(u) = s(u, u) > 0 \qquad \text{for all non-zero } u \in H. \tag{5}$$

We now give the definition of an *inner product* using sesquilinear forms.

Definition 2 (Inner product)**.** *Let H be a vector space over $\mathbb{F}$. An* inner product *on H is a positive, Hermitian, sesquilinear form.*

An inner product is denoted by $\langle \cdot, \cdot \rangle$.

Definition 3 (Inner product space)**.** *If H is a (real or complex) vector space and $\langle \cdot, \cdot \rangle$ is an inner product on H, then the pair $(H, \langle \cdot, \cdot \rangle)$ is called an* inner product space *or a* pre-Hilbert space.

Theorem 1 (Schwarz inequality)**.** *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. For every $u, v \in H$ we have the* Schwarz inequality

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle, \tag{6}$$

with the equality holding if and only if u and v are linearly dependent.

Proof. Let $u, v \in H$. For all $t \in \mathbb{R}$, we have

$$\begin{aligned} 0 \leq \langle u + tv, u + tv \rangle &= \langle u, u \rangle + t \langle v, u \rangle + t^* \langle u, v \rangle + t^2 \langle v, v \rangle \\ &= \langle u, u \rangle + 2t \operatorname{Re}\langle u, v \rangle + t^2 \langle v, v \rangle, \end{aligned} \tag{7}$$

which is a second degree polynomial in t. Non-negativity implies that it either has no root or a double root. For a polynomial $at^2 + bt + c$, this holds when $b^2 - 4ac \leq 0$, and it follows that

$$4[\operatorname{Re}\langle u, v \rangle]^2 - 4\langle u, u \rangle \langle v, v \rangle \leq 0 \implies [\operatorname{Re}\langle u, v \rangle]^2 \leq \langle u, u \rangle \langle v, v \rangle^2. \tag{8}$$

If we choose $a \in \mathbb{F}$ such that $a\langle u, v \rangle = |\langle u, v \rangle|$, we have $|a| = 1$ and

$$\begin{aligned} |\langle u, v \rangle|^2 &= [\operatorname{Re} a\langle u, v \rangle]^2 = [\operatorname{Re}\langle au, v \rangle]^2 \\ &\leq \langle au, au \rangle \langle v, v \rangle = |a|^2 \langle u, u \rangle \langle v, v \rangle = \langle v, v \rangle \langle u, u \rangle; \end{aligned} \tag{9}$$

the Schwarz inequality.

Now we consider the case when the equality $|\langle u,v\rangle|^2 = \langle u,u\rangle\langle v,v\rangle$ holds. Because of this, the polynomial considered above has a double root, and we have $t_0 \in \mathbb{R}$ such that $\langle u + t_0 v, u + t_0 v\rangle = 0$, i.e., $u = -t_0 v$.

Conversely, if u and v are linearly dependent, i.e., $u = av$ for some $a \in \mathbb{F}$, then

$$\begin{aligned} |\langle u,v\rangle|^2 &= |\langle av,v\rangle|^2 = \langle av,v\rangle\langle v,av\rangle \\ &= \langle av,av\rangle\langle v,v\rangle = \langle u,u\rangle\langle v,v\rangle. \end{aligned} \tag{10}$$

□

Definition 4 (Norm). *A mapping $p : H \to \mathbb{R}$ is called a* norm *on H if for all $u, v \in H$ and $a \in \mathbb{F}$ we have*

1. *$p(u) \geq 0$ and $p(u) = 0 \iff u = 0$.*
2. *$p(au) = |a|p(u)$, and*
3. *$p(u+v) \leq p(u) + p(v)$ (triangle inequality).*

A norm is denoted by $||\cdot||$.

Definition 5 (Normed space). *If H is a (real or complex) vector space and $||\cdot||$ is an inner product on H, then the pair $(H, ||\cdot||)$ is called a* normed space.

Schwarz inequality has the following straightforward application.

Proposition 1. *If $(H, \langle\cdot,\cdot\rangle)$ is a (real or complex) inner product space, then $||u|| = \langle u,u\rangle^{1/2}$ defines a norm on H.*

Proof. As the inner product is defined to be a *positive* sesquilinear form, it follows that for every $u \in H$, $||u|| \geq 0$ and $||u|| = 0 \iff u = 0$.

For any $u \in H, a \in \mathbb{F}$, we have

$$\begin{aligned} ||au|| &= \langle au, au\rangle^{1/2} \\ &= [|a|^2\langle u,u\rangle]^{1/2} \\ &= |a|\langle u,u\rangle^{1/2} \\ &= |a|||u||. \end{aligned}$$

For the triangle inequality, consider $u, v \in H$ and

$$\begin{aligned}
||u+v||^2 &= \langle u+v, u+v\rangle \\
&= \langle u,u\rangle + 2\,\mathrm{Re}\langle u,v\rangle + \langle v,v\rangle \\
&= ||u||^2 + 2\,\mathrm{Re}\langle u,v\rangle + ||v||^2 \\
&\leq ||u||^2 + 2|\langle u,v\rangle| + ||v||^2 \\
&\leq ||u||^2 + 2||u||||v|| + ||v||^2 \\
&= (||u|| + ||v||)^2.
\end{aligned}$$

Schwarz inequality was used in the fifth line. □

1.2 Topological Notions

Definition 6 (Convergence). *Let $(H, ||\cdot||)$ be a normed space. A sequence (u_n) in H is said to converge to $u \in H$ if, given arbitrary $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $||u_n - u|| < \epsilon$, whenever $n > N$.*

A sequence (u_n) is said to be *convergent* if there exists a $u \in H$ such that (u_n) converges to u. We recall that convergent sequences are *bounded*.

Definition 7 (Cauchy sequence). *A sequence (u_n) is said to be a* Cauchy sequence *if, given arbitrary $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $||u_n - u_m|| < \epsilon$ whenever $n, m > N$.*

Proposition 2. *If (u_n) is a Cauchy sequence, then the sequence $(||u_n||)$ is convergent.*

Proof. Let $\epsilon > 0$ be given. Since (u_n) is Cauchy, there exists $N \in \mathbb{N}$ such that $||u_n - u_m|| < \epsilon$ when $n, m > N$.

$$|||u_n|| - ||u_m||| \leq ||u_n - u_m|| < \epsilon. \tag{11}$$

Hence $(||u_n||)$ is a Cauchy sequence in $\mathbb{R}$, and therefore converges. □

Proposition 3. *If $(H, \langle\cdot,\cdot\rangle)$ is an inner product space and (u_n) and (v_n) are Cauchy sequences, then the sequence $(\langle u_n, v_n\rangle)$ is convergent.*

Proof. Let $\epsilon > 0$ be given. Since $(u_n), (v_n)$ are Cauchy, $(||u_n||), (||v_n||)$ are convergent and therefore bounded. Let $0 < C < \infty$ be such that $||v_n||, ||u_n|| < C$, for every $n \in \mathbb{N}$. Since $(u_n), (v_n)$ are Cauchy, there exists $N \in \mathbb{N}$ such that $||u_n - u_m|| < \epsilon/2C$ and $||v_n - v_m|| < \epsilon/2C$, whenever $n, m > N$.

Finally, consider

$$\begin{aligned}
|\langle u_n, v_n\rangle - \langle u_m, v_m\rangle| &= |\langle u_n, v_n\rangle - \langle u_m, v_n\rangle + \langle u_m, v_n\rangle - \langle u_m, v_m\rangle| \\
&= |\langle u_n - u_m, v_n\rangle + \langle u_m, v_n - v_m\rangle| \\
&\leq |\langle u_n - u_m, v_n\rangle| + |\langle u_m, v_n - v_m\rangle| \\
&\leq ||u_n - u_m||||v_n|| + ||u_m||||v_n - v_m|| \\
&< \frac{\epsilon}{2C}\cdot C + C\cdot\frac{\epsilon}{2C} = \epsilon,
\end{aligned} \tag{12}$$

whenever $m,n > N$. $(\langle u_n, v_n \rangle)$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges. □

Note that if we take (v_n) to be a constant sequence with $v_n = v$ for every $n \in \mathbb{N}$, the following result comes as a corollary.

Proposition 4. *If H is an inner product space, v is an arbitrary element of H and (u_n) is a Cauchy sequence, then the sequence $(\langle u_n, v\rangle)$ is convergent.*

Definition 8 (Completeness). *A normed space $(H, ||\cdot||)$ is said to be* complete *if every Cauchy sequence is convergent.*

A complete normed space is called a *Banach space*. A complete inner product space is called a *Hilbert space*.

In addition to sequences, the norm allows us to talk of open and closed subsets in H. Open balls in $(H, ||\cdot||)$ are defined in the usual way.

Definition 9 (Open ball). *An open ball of radius $r > 0$, centred at $u \in H$ is defined as the set*

$$B(u,r) = \{v \in H : ||v - u|| < r\}. \tag{13}$$

The collection of all open balls is a basis for the usual topology in H. With respect to this topology, the notion of closed sets and closure of subsets is clear. Closure of a subset A, denoted $\overline{A}$, is the collection of all limit points of A.

Theorem 2. *Closure of a subspace of H is a subspace.*

Proof. Let T be a subspace of H. Let $u, v \in \overline{T}$ and let $a, b \in \mathbb{F}$. Since u, v are limit points of T, there are sequences (u_n) and (v_n) which converge to u and v respectively. It follows that

$$au + bv = a \lim u_n + b \lim v_n = \lim(au_n + bv_n). \tag{14}$$

Since, $au_n + bv_n \in T$, it follows that $au + bv \in \overline{T}$. □

Theorem 3. *A subspace T of a Banach (Hilbert) space H is closed if and only if T is a Banach (Hilbert) space.*

Proof. Assume that T is closed. If (u_n) is a Cauchy sequence in T, there exists $u \in H$ such that $\lim u_n = u$, because H is complete. But, u is a limit point of T and since, by hypothesis, T is closed, $u \in T$. Hence, T is complete and therefore a Banach (Hilbert) space.

Conversely, assume that T is a Banach (Hilbert) space. If $u \in \overline{T}$, then there exists a sequence (u_n) in T such that $\lim u_n = u$. As (u_n) is convergent in H, it is Cauchy, and if it is Cauchy, it must converge in T, i.e., $u \in T$. T is closed. □

Another important topological concept is that of *dense subsets*.

Definition 10. *Let A and B be subsets of a normed space H. The set A is said to be* dense relative *to B if $B \subset \overline{A}$ holds. If, in addition, $A \subset B$, then we say that A is a* dense subset *of B. If A is dense relative to H, then we say briefly that A is* dense.

Definition 11. *A subset A of a normed space is said to be* separable *if there exists a countable subset B of A which is dense in A.*

The finite dimensional space $\mathbb{R}^n$ and $\mathbb{C}^n$ are separable, as the set of vectors with rational components is countable and dense.

Definition 12 (Span)**.** *Let B be a subset of an inner product space. The* span *of B, denoted* span$B$ *is the set of all finite linear combinations of elements of B.*

Span of B is the smallest subspace of H containing B.

Definition 13. *A set B is said to be* total with respect to T *if the span of B is dense relative to T. The set B is said to be* total in T *if $B \subset T$ and it is said to be* total *if $T = H$.*

1.3 Orthogonality

Definition 14 (Orthogonal)**.** *Two elements of an inner product space are called* orthogonal *if their inner product vanishes.*

We have the following simple result in context of inner product spaces.

Proposition 5 (Pythagoras's theorem)**.** *If u and v are orthogonal elements of an inner product space, then*

$$||u+v||^2 = ||u||^2 + ||v||^2. \tag{15}$$

Proof. We have

$$\begin{aligned} ||u+v||^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= ||u||^2 + ||v||^2, \end{aligned}$$

as required. □

The above result can be generalised to a finite, set of mutually orthogonal elements of an inner product space, i.e., $A = \{u_1, \ldots, u_n\}$ with $\langle u_i, u_j\rangle = 0$ whenever $i \neq j$. The proof involves induction over n.

Theorem 4 (Generalised Pythagoras's theorem)**.** *If $u_1, \ldots, u_n$ is a finite set of mutually orthogonal elements of an inner product space, then the following result holds,*

$$\left\| \sum_{j=1}^{n} u_j \right\|^2 = \sum_{j=1}^{n} \left\| u_j \right\|^2. \tag{16}$$

Definition 15 (Orthogonal complement)**.** *Let H be an inner product space. If A is a subset of H, then the set $A^\perp = \{u \in H : \langle u, v\rangle = 0$ for every $v \in A\}$ is called the* orthogonal complement *of* A.

The notation $u \perp A$ is often used if $\langle u, v\rangle = 0$ for every $v \in A$.

In the following string of propositions, let H be an inner product space.

Proposition 6. *We have* $\{0\}^\perp = H$ *and* $H^\perp = \{0\}$*, i.e.,* 0 *is the only element orthogonal to every element.*

Proof. For every $u \in H$ we have $\langle u, 0\rangle = 0$, therefore $\{0\}^\perp = H$. If $u \neq 0$, then $\langle u, u\rangle \neq 0$, i.e., u is not orthogonal to H; hence $H^\perp = \{0\}$. □

Proposition 7. *For every subset A of H, the set* $A^\perp$ *is a closed subspace of* H*.*

Proof. If $u, v \in A^\perp$ and $a, b \in \mathbb{F}$, then for all $w \in A$, we have

$$\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle = 0, \tag{17}$$

and therefore $au + bv \in A^\perp$ is a subspace.

To show that $A^\perp$ is closed, let $u \in \overline{A^\perp}$, and let (u_n) be a sequence in $A^\perp$ such that $\lim u_n = u$. For every $w \in A$, Proposition 3 gives us

$$\langle u, v\rangle = \lim\langle u_n, v\rangle = 0, \tag{18}$$

and $u \in A^\perp$. $A^\perp$ is closed. □

Proposition 8. $A \subset B$ *implies* $B^\perp \subset A^\perp$.

Proof. If $u \in B^\perp$, we have $\langle u, v\rangle = 0$ for every $v \in B$. However, since $A \subset B$, we also have $\langle u, v\rangle = 0$ for every $v \in A$, and we have $u \in A^\perp$. □

Proposition 9. *We have* $A^\perp = \operatorname{span} A^\perp = \overline{\operatorname{span} A}^\perp$.

Proof. Since $A \subset \operatorname{span} A \subset \overline{\operatorname{span} A}$, from Proposition 8 we have $\overline{\operatorname{span} A}^\perp \subset \operatorname{span} A^\perp \subset A^\perp$.

If $u \in A^\perp$, then $\langle u, w, =\rangle 0$ for all $w \in A$, and therefore $w \in \operatorname{span} A$. If $w \in \overline{\operatorname{span} A}$, then there exists a sequence (w_n) in $\operatorname{span} A$ such that $\lim w_n = w$, and consequently, $\langle u, w\rangle = \lim\langle u, w_n\rangle = 0$, and hence $u \in \overline{\operatorname{span} A}^\perp$. □

We also want to state the projection theorem, but defer the proof.

Theorem 5 (Projection theorem)**.** *Let H be a Hilbert space, and let T be a closed spbspace of H. Then we have* $T^{\perp\perp} = T$*. Each* $u \in H$ *can be uniquely decomposed in the form* $u = v + w$ *with* $v \in T$ *and* $w \in T^\perp$*.*

The projection theorem has (among many others) the following two consequences.

Proposition 10. *Let H be a Hilbert space. For every subset A of H we have* $A^{\perp\perp} = \overline{\operatorname{span} A}$*.*

Proof. Since $A^\perp = \overline{\operatorname{span} A}^\perp$, the projection theorem shows that $\overline{\operatorname{span} A} = \overline{\operatorname{span} A}^{\perp\perp} = A^{\perp\perp}$. ☐

Proposition 11. *In a Hilbert space H, we have $A^\perp = \{0\}$ if and only if A is total, i.e., $A^\perp = \{0\}$ if and only if $\overline{\operatorname{span} A} = H$ holds.*

Proof. If $A^\perp = \{0\}$, then we have $\overline{\operatorname{span} A} = A^{\perp\perp} = \{0\}^\perp = H$. Conversely, if $\overline{\operatorname{span} A} = A^{\perp\perp} = H$, then, since $A^\perp$ is a closed subspace, we have $A^\perp = A^{\perp\perp\perp} = H^\perp = \{0\}$. ☐

If T_1 and T_2 are subspaces of a vector space such that $T_1 \cap T_2 = \{0\}$, then $T_1 \oplus T_2 = \{u + v : u \in T_1, v \in T_2\}$ is a *direct sum*, i.e. each element of $T_1 \oplus T_2$ has a unique representation of the form $u + v$ with $u \in T_1$ and $v \in T_2$.

If T_1 and T_2 are subspaces of an inner product space with $T_1 \perp T_2$, then we have $T_1 \cap T_2 = \{0\}$. In this case the direct sum $T_1 + T_2$ is called an *orthogonal sum* and is denoted by $T_1 \hat{\oplus} T_2$.

Orthogonal sums have the following topological properties.

Proposition 12. *Let H be an inner product space, and let T_1 and T_2 be orthogonal subspaces. If $T_1 \hat{\oplus} T_2$ is closed, then T_1 and T_2 are closed.*

The proof proceeds by choosing an element $u \in \overline{T_j}$, a sequence (u_n) in T_j that converges to u, and using the fact that $T_1 \hat{\oplus} T_2$ is closed to show that $u \in T_j$. $j = 1, 2$.

Proposition 13. *If H is a Hilbert space and T_1 and T_2 are closed orthogonal subspaces, then $T_1 \hat{\oplus} T_2$ is closed.*

The proof proceeds by choosing an element $w \in \overline{T_1 \hat{\oplus} T_2}$, a sequence $(u_n + v_n)$ in $T_1 \hat{\oplus} T_2$ such that $u_n \in T_1$ and $v_n \in T_2$, and using the fact that T_1 and T_2 are closed to show $w \in T_1 \hat{\oplus} T_2$.

1.4 Orthonormal Bases

Definition 16. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space. A family $M = \{e_\alpha : \alpha \in A\}$ of elements from H is called an* orthonormal system *if we have $\langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta}$ for every $\alpha, \beta \in A$, where $\delta_{\alpha\beta}$ is the Kronecker delta.*

Hence, in addition to being orthogonal, elements of an orthonormal system are *normalized*, i.e., $||e|| = 1$ for every $e \in M$.

An orthonormal system is said to be *linearly independent* if each of its finite subsystems is linearly independent.

Proposition 14. *Each orthonormal system is linearly independent.*

Proof. Let an orthonormal system be given; consider an arbitrary finite subsystem $\{e_1, \ldots, e_n\}$ of the orthonormal system. This subsystem is also an orthonormal system.

If $\sum_{j=1}^{n} a_j e_j = 0$, then

$$0 = \left\| \sum_{j=1}^{n} a_j e_j \right\|^2 = \sum_{j=1}^{n} \left| a_j \right|^2 \tag{19}$$

follows from generalised Pythagoras's theorem. However, since each $\left| a_j \right|$ is non-negative, the only possibility is $a_j = 0$ for every j. □

Definition 17. *Let H be a Hilbert space. An orthonormal system M is called an* orthonormal basis *of H if M is total in H.*

It follows from the above definition that an orthonormal system M is an orthonormal basis of the Hilbert space if and only if $H = \overline{\operatorname{span} M}$.

An orthonormal system M is said to be *maximal* if for every orthonormal system M' with $M \subset M'$, the relation $M' = M$ holds.

Proposition 15. *Each orthonormal basis is a maximal orthonormal system.*

Proof. Let M be an orthonormal basis. Assume to the contrary that M is not maximal: there exists an $e \in H$ such that $e \perp M$, which implies $e \perp \overline{\operatorname{span} M}$. This contradicts $\overline{\operatorname{span} M} = H$. □

Proposition 16. *If H is a Hilbert space, then each maximal orthonormal system is an orthonormal basis.*

Proof. Let M be a maximal orthonormal system in H. If M were not an orthonormal basis, i.e., if M were not total in H, there would exist an $e \in H$ such that $e \perp \operatorname{span} M$ and $||e|| = 1$. Hence, $M' = M \cup \{e\}$ would be a larger orthonormal system, contradicting the maximality of M. □

We now want to prove the Bessel inequality and Parseval identity for Hilbert spaces. But first, we need the following result.

For any $u \in H$ and $A \subset H$, the distance between u and A is defined as

$$d(u, A) = \inf\{d(u, w) : w \in A\}. \tag{20}$$

Lemma 1. *Let H be an inner product space. If $\{e_1, \ldots, e_n\}$ is a finite orthonormal system in H, then for each $u \in H$ there exists a $v \in \operatorname{span}(e_1, \ldots, e_n)$ such that $||u - v|| = d(u, \operatorname{span}(e_1, \ldots, e_n))$, and we have*

$$v = \sum_{j=1}^{n} \langle u, e_j \rangle e_j. \tag{21}$$

Proof. For all $c_1,\ldots,c_n \in \mathbb{F}$ we have

$$\begin{aligned}\left\|u-\sum_{j=1}^{n}c_je_j\right\|^2 &= \langle u-\sum_{j=1}^{n}c_je_j, u-\sum_{j=1}^{n}c_je_j\rangle \\ &= \|u\|^2-\sum_{j=1}^{n}c_j\langle e_j,u\rangle-\sum_{j=1}^{n}c_j^*\langle u,e_j\rangle+\sum_{j=1}^{n}\left|c_j\right|^2 \\ &= \|u\|^2-\sum_{j=1}^{n}\left|\langle u,e_j\rangle\right|^2+\sum_{j=1}^{n}\left|c_j-\langle u,e_j\rangle\right|^2, \end{aligned} \tag{22}$$

which is minimized when $c_j = \langle u,e_j\rangle$ for every $j=1,\ldots,n$. Hence, we have $v = \sum_{j=1}^{n} = \langle u,e_j\rangle e_j \in \mathrm{span}(e_1,\ldots e_n)$ such that

$$d(u,\mathrm{span}(e_n,\ldots,e_n)) = \|u-v\|. \tag{23}$$

□

Theorem 6 (Bessel inequality). *Let $\{e_\alpha : \alpha\in A\}$ be an orthonormal system in H and let $u\in H$. Then at most countably many of the numbers $\langle u,e_\alpha\rangle$ are non-zero, and we have the Bessel inequality*

$$\|u\|^2 \geq \sum_{\alpha\in A}|\langle u,e_\alpha\rangle|^2. \tag{24}$$

Proof. For every finite set $\{\alpha_1,\ldots,\alpha_n\}\subset A$ we have

$$\|u\|^2 = \left\|u-\sum_{j=1}^{n}\langle u,e_{\alpha_j}\rangle e_{\alpha_j}\right\|^2+\sum_{j=1}^{n}\left|\langle u,e_{\alpha_j}\rangle\right|^2 \tag{25}$$

by Lemma 1. Hence

$$\sum_{j=1}^{n}\left|\langle u,e_{\alpha_j}\rangle\right|^2 \leq \|u\|^2. \tag{26}$$

Since, the sum above is bounded above for every finite subset of A, it implies that the sum $\sum_{\alpha\in A}|\langle u,e_\alpha\rangle|^2$ converges, and the Bessel inequality follows.

Since, the summation is of non-negative real numbers and it converges, there are only countably many $\alpha\in A$ such that $\langle u,e_\alpha\rangle$ is non-zero. □

Theorem 7 (Parseval identity). *An orthonormal system $M=\{e_\alpha : \alpha\in A\}$ is a basis if and only if for all $u\in H$ the Parseval identity*

$$\|u\|^2 = \sum_{\alpha\in A}|\langle u,e_\alpha\rangle|^2 \tag{27}$$

holds. We also have

$$u = \sum_{\alpha\in A}\langle u,e_\alpha\rangle e_\alpha \qquad \textit{for every } u\in H. \tag{28}$$

Proof. If the Parseval identity holds for every $u \in H$, then there is a sequence (α_j) of elements of A for which $\langle u, e_\alpha \rangle \neq 0$. Then, by Lemma 1 we have

$$\left\| u - \sum_{j=1}^{n} \langle u, e_{\alpha_j} \rangle e_{\alpha_j} \right\|^2 = ||u||^2 - \sum_{j=1}^{n} \left| \langle u, e_{\alpha_j} \rangle \right|^2. \tag{29}$$

Due to Parseval identity, $\lim \sum_{j=1}^{n} |\langle u, e_{\alpha_j} \rangle|^2 = ||u||^2$, and RHS of the above equation goes to 0 as $n \to \infty$. Consequently, $u \in \overline{\operatorname{span} M}$ and

$$u = \sum_{j=1}^{\infty} \langle u, e_{\alpha_j} \rangle e_{\alpha_j} = \sum_{\alpha \in A} \langle u, e_\alpha \rangle e_\alpha, \tag{30}$$

and we conclude that M is an orthonormal basis.

Conversely, assume that M is an orthonormal basis, $H = \overline{\operatorname{span} M}$. For every $\epsilon > 0$ and $u \in H$, there exists $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in A$ and $c_1, \ldots, c_n \in \mathbb{F}$ such that

$$\left\| f - \sum_{j=1}^{n} c_j e_{\alpha_j} \right\|^2 < \epsilon. \tag{31}$$

From Bessel inequality and Lemma 1, it follows that

$$\begin{aligned} 0 &\leq ||u||^2 - \sum_{\alpha \in A} |\langle u, e_\alpha \rangle|^2 \\ &\leq ||u||^2 - \sum_{j=1}^{n} \left| \langle u, e_{\alpha_j} \rangle \right|^2 = \left\| u - \sum_{j=1}^{n} \langle u, e_{\alpha_j} \rangle e_{\alpha_j} \right\|^2 \\ &\leq \left\| u - \sum_{j=1}^{n} c_j e_{\alpha_j} \right\|^2 < \epsilon, \end{aligned} \tag{32}$$

and since ϵ is arbitrary

$$||u||^2 = \sum_{\alpha \in A} |\langle u, e_\alpha \rangle|^2. \tag{33}$$

□

1.5 Existence of a Basis

We have seen some properties of orthonormal bases, but we still need to prove that a basis exists for all inner product spaces or Hilbert spaces. For separable inner product spaces, it is easy to show that an orthogonal basis exists. For non-separable spaces the proof is a little harder and involves an invocation of choice (in the form of Zorn's lemma).

Theorem 8. *Let H be a separable inner product space. If M_0 is a finite orthonormal system in H, then there exists an orthonormal basis M in H such that $M \supset M_0$.*

Proof. Let H be a non-trivial space, let $M_0 = \{e_1, \ldots, e_n\}$ be a finite orthonormal system. Since, H is separable, there is a countable subset $A = u_n : n \in \mathbb{N}$ dense in H.

Define the elements $g_1, g_2, \ldots$ inductively: let $g_1 = f_{j_1}$, where j_1 is the smallest index for which $\{e_1, \ldots, e_n, f_{j_1}\}$ is linearly independent. If $g_1, \ldots, g_k$ are defined, let $g_{k+1} = f_{j_{k+1}}$, where j_{k+1} is the smallest index for which $\{e_1, \ldots, e_n, g_1, \ldots, g_k, f_{j_{k+1}}\}$ is linearly independent.

With $B = \{e_1, \ldots, e_n, g_1, g_2, \ldots\}$ we have $\operatorname{span} A \subset \operatorname{span} B$; since A is dense, B is total. To obtain an orthonormal basis M, we can apply Gram–Schmidt orthogonalization to B; the first n elements of M will coincide with $e_1, \ldots, e_n$ as these are already orthonormal. We have $\overline{\operatorname{span} M} = \overline{\operatorname{span} B} = H$. Hence M is an orthonormal basis with $M_0 \subset M$. □

Existence of an orthonormal basis for a separable inner product space follows directly from Theorem 8 by taking $M_0 = \emptyset$.

Theorem 9. *A separable inner product space possesses an orthonormal basis.*

The next result tells us that dimension of a finite-dimensional, inner product space is well defined.

Theorem 10. *Let H be a separable inner product space. H is m-dimensional ($m < \infty$) if and only if there exists an orthonormal basis containing m elements.*

Proof. Let H be m-dimensional; assume that the maximal number of linearly independent elements equals m. As every orthonormal system is linearly independent (Proposition 14), it consists of at most m elements. If $M = \{e_1, \ldots, e_k\}$ is an orthonormal system with less than m elements, then $\dim \operatorname{span} M < \dim H$ and there exists an $f \in H$ such that $\{e_1, \ldots, e_n, f\}$ is linearly independent. Gram–Schmidt orthogonalization gives a system $M' = \{e_1, \ldots, e_k, e_{k+1}\}$ such that $M \subset M'$; M is not an orthonormal basis. Each orthonormal basis must have exactly m elements. □

Now, we want an analogous result for infinite dimensional spaces.

Theorem 11. *Let H be a separable inner product space. H is infinite dimensional if and only if there exists an orthonormal basis with countably infinite elements. Each orthonormal basis in H is countably infinite.*

Proof. If H is infinite dimensional, then each basis must have at least countably infinite elements, for otherwise H would be finite dimensional. It suffices to show that every orthonormal basis $M = \{e_\alpha : \alpha \in A\}$ is countable.

Let $N = \{f_n : n \in \mathbb{N}\}$ be a countable dense subset. For each $\alpha \in A$ there exists an $n(\alpha) \in \mathbb{N}$ such that $||f_{n(\alpha)} - e_\alpha|| < 1/2$. Because e_α and e_β are normalised and orthogonal if $\alpha \neq \beta$, we have $\left\|e_\alpha - e_\beta\right\| = [||e_\alpha|| + \left\|-e_\beta\right\|]^{1/2} = \sqrt{2}$ and

$$\begin{aligned}\left\|f_{n(\alpha)} - f_{n(\beta)}\right\| &\geq \left\|e_\alpha - e_\beta\right\| - \left\|f_{n(\alpha)} - e_\alpha\right\| - \left\|e_\beta - f_{n(\beta)}\right\| \\ &\geq \sqrt{2} - 1 > 0 \qquad \text{for } \alpha \neq \beta.\end{aligned}$$

This implies that the mapping $\alpha \mapsto n(\alpha) : A \to \mathbb{N}$ is injective; A is countable. □

Proposition 17. *An inner product space is separable if and only if it possesses a countable orthonormal basis.*

Proof. By Theorem 10–11, a separable inner product space possesses an at most countable orthonormal basis. Conversely, if M is a countable orthonormal basis in H, then the $\text{span}_r M$ of finite linear combinations of elements of M with rational coefficients is dense in $\text{span} M$, and thus is also dense in H. As $\text{span}_r M$ is at most countable, H is separable. □

We now proceed to prove a version of Theorem 8 for a general Hilbert space which is not necessarily separable. For infinite dimensional space, the proof requires an application of *Zorn's lemma*, which we now state.

Theorem 12 (Zorn's lemma)**.** *Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.*

Now, on to the existence of a basis.

Theorem 13. *Let H be a Hilbert space. If M_0 is an orthonormal system, then there exists an orthonormal basis M in H such that $M \supset M_0$.*

We shall prove this result in a string of lemmas.

Lemma 2. *Let $\mathcal{M}$ be the collection of all orthonormal systems which contain M_0. $\mathcal{M}$ is partially ordered by the inclusion "$\subset$".*

Proof. We have to show that the relation defined by inclusion is reflexive, antisymmetric, and transitive. For every $M \in \mathcal{M}$ we have $M \subset M$ (reflexive). For $M_1, M_2 \in \mathcal{M}$, if $M_1 \subset M_2$ and $M_2 \subset M_1$, then $M_1 = M_2$ (antisymmetric). If $M_1, M_2, M_3 \in \mathcal{M}$ such that $M_1 \subset M_2$ and $M_2 \subset M_3$, it follows that $M_1 \subset M_3$ (transitive). Hence, inclusion induces a partial order on $\mathcal{M}$. □

Lemma 3. *If $\mathcal{N}$ is a chain in $\mathcal{M}$, i.e., if $\mathcal{N}$ is a totally ordered subset of $\mathcal{M}$, then $\mathcal{N}$ has an upper bound.*

Proof. For the upper bound M, we take the union of all $N \in \mathcal{N}$. We have to show that $M \in \mathcal{M}$, i.e., M is an orthonormal system. If $u_1, u_2 \in M$, then there exist $M_1, M_2 \in \mathcal{N}$ such that $u_1 \in M_1$ and $u_2 \in M_2$. Since $\mathcal{N}$ is totally ordered, either $M_1 \subset M_2$ or $M_2 \subset M_1$ holds, i.e., either $u_1, u_2 \in M_1$ or $u_1, u_2 \in M_2$. Therefore $u_1 \perp u_2$. □

Proof of Theorem 13. Lemma 2–3 and Zorn's lemma imply the existence of at least one *maximal element* $M_{\text{max}} \in \mathcal{M}$, such that for each $M \in \mathcal{M}$ satisfying $M_{\text{max}} \subset M$, we have $M_{\text{max}} = M$.

Finally, we have to show that M_{max} is an orthonormal basis. If $\overline{\text{span} M_{\text{max}}} \neq H$, then by Proposition 11, there exists a $u \perp \text{span} M_{\text{max}}$ such that $||u|| = 1$. In such a case $M_{\text{max}} \cup \{u\}$ would be an orthonormal system such that $M_{\text{max}} \subset M_{\text{max}} \cup \{u\}$ and

$M_{\max} \neq M_{\max} \cup \{u\}$; a contradiction to the maximality of $M_{\max}$. The requirement $M_0 \subset M_{\max}$ is satisfied by construction. □

To end this section, we want to prove analogues of Theorems 10–11 for a Hilbert space that is not necessarily separable.

Theorem 14. *All orthonormal bases of a Hilbert space have the same cardinality.*

Proof. Let M_1 and M_2 be orthonormal bases of H. If $|M_1| = m < \infty$ (denote by $|M|$ the cardinality of M), then by Proposition 17, H is separable and by Theorem 10 we have $\dim H = |M_1| = |M_2| = m$.

Now let M_1 be infinite. For each $u \in M_1$, let $K(u) = \{v \in M_2 : \langle u, v\rangle \neq 0\}$; by arguments involved in the proof of the Bessel inequality (Theorem 6), $K(u)$ is at most countable. We claim that $\cup_{u\in M_1} K(u) = M_2$, for otherwise $v \in M_2 \setminus U_{u\in M_1} K(u)$ would imply $v \perp M_1$, therefore $v = 0$ as M_1 is total. However, this is impossible, since all elements of M_1 have unit norm. Consequently, it follows that

$$|M_2| \leq \sum_{u\in M_1} |K(u)| \leq |M_1||\mathbb{N}| \leq |M_1|. \tag{34}$$

Completely analogously, switching the roles of M_1 and M_2 in the above construction leads to $|M_1| \leq |M_1|$. Hence, $|M_1| = |M_2|$. □

Theorem 14 shows that the *Hilbert space dimension*, defined as the cardinality of an orthonormal basis does not depend on the choice of orthonormal basis. Theorem 10 shows that the Hilbert space dimension coincides with the *algebraic dimension* (cardinality of a maximal linearly independent set) in case of finite-dimensional spaces.

1.6 Tensor Product of Hilbert Spaces

Let H_1 and H_2 be vector spaces over $\mathbb{F}$. Denote by $F(H_1, H_2)$, the vector space of all *formal* linear combinations of pairs (u, v) with $u \in H_1$ and $v \in H_2$:

$$F(H_1, H_2) = \left\{ \sum_{j=1}^{n} c_j(u_j, v_j) : c_j \in \mathbb{F}, u_j \in H_1, v_j \in H_2, j = 1, \ldots, n; n \in \mathbb{N} \right\}$$

If we denote the pair (u, v) by $u \otimes v$, then $F(H_1, H_2)$ consists of elements of the form

$$\sum_{j=1}^{n} c_j(u_j \otimes v_j), \tag{35}$$

for $c_j \in \mathbb{F}, u_j \in H_1, v_j \in H_2$. However, we cannot identify $F(H_1, H_2)$ as the tensor product space just yet. We want the symbol ⊗ to behave like a "product", i.e., it should "distribute" over vector addition and "commute" with scalar multiplication. In particular, for $a_j, b_k \in \mathbb{F}$; $u_j \in H_1$; $v_k \in H_2$; and $j = 1, \ldots, n$; $k = 1, \ldots, m$ we should have

$$\left(\sum_{j=1}^{n} a_j u_j\right) \otimes \left(\sum_{k=1}^{m} b_k v_k\right) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j b_k (u_j \otimes v_k). \tag{36}$$

Hence, if we consider the subspace N of $F(H_1, H_2)$ spanned by elements of the form

$$\left(\sum_{j=1}^{n} a_j u_j\right) \otimes \left(\sum_{k=1}^{m} b_k v_k\right) - \sum_{j=1}^{n}\sum_{k=1}^{m} a_j b_k (u_j \otimes v_k), \tag{37}$$

defining the quotient space $F(H_1, H_2)/N$ as the *algebraic tensor product* places LHS and RHS of (36) in the same equivalence class. We denote $F(H_1, H_2)/N$ by $H_1 \otimes H_2$.

If we think of the vector space as an abelian group under addition, the quotient vector space is identical to the quotient group, and elements whose difference is in N are put in the same equivalence class.

The product $H_1 \times H_2$ can be considered a subset of $F(H_1, H_2)$, if we identify the pair (u, v) as $u \otimes v \in F(H_1, H_2)$. In a slight abuse of notation, the equivalence class in $H_1 \otimes H_2$ containing $u \otimes v$ is also denoted by $u \otimes v$; these elements are called *simple tensors*. A linear combination of simple tensors is zero if and only if it is a finite linear combination of elements of the form

$$\sum_{j=1}^{n}\sum_{k=1}^{m} a_j b_k (u_j \otimes v_k) - \left(\sum_{j=1}^{n} a_j u_j\right) \otimes \left(\sum_{k=1}^{m} b_k v_k\right). \tag{38}$$

If $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ are Hilbert spaces over $\mathbb{F}$, then

$$\left\langle \sum_{j=1}^{n} c'_j u'_j \otimes v'_j, \sum_{k=1}^{m} c_k u_k \otimes v_k \right\rangle = \sum_{j=1}^{n}\sum_{k=1}^{m} c'_j c^*_k \langle u'_j, u_k\rangle_1 \langle v'_j, v_k\rangle_2 \tag{39}$$

defines a Hermitian sesquilinear form on $H_1 \otimes H_2$. To show that $\langle\cdot,\cdot\rangle$ is an inner product on $H_1 \otimes H_2$ it suffices to show that $\langle u, u\rangle > 0$ holds for all non-zero $f \in H_1 \otimes H_2$. Let $u = \sum_{j=1}^{n} c_j u_j \otimes v_j \neq 0$. If $\{e_k\}$ and $\{e'_k\}$ are orthonormal bases of $\mathrm{span}(u_1, \ldots, u_n)$ and $\mathrm{span}(v_1, \ldots, v_n)$ respectively, then

$$u = \sum_j \sum_k \sum_l c_j \langle u_j, e_k\rangle \langle v_j, e'_l\rangle e_k \otimes e'_l = \sum_{k,l} c_{kl} e_k \otimes e'_l, \tag{40}$$

where $c_{kl} = \sum_j c_j \langle u_j, e_k\rangle \langle v_j, e'_l\rangle$ and thus

$$\langle u, u\rangle = \sum_{k,l} |c_{kl}|^2 > 0. \tag{41}$$

Therefore, $(H_1 \otimes H_2, \langle\cdot,\cdot\rangle)$ is an inner product space. The completion of this inner product space is denoted $H_1 \hat{\otimes} H_2$ and is called the *(complete) tensor product* of Hilbert spaces H_1 and H_2.

Proposition 18. *Let H_1 and H_2 be inner product spaces. For subsets $M_1 \subset H_1$ and $M_2 \subset H_2$, the relation* $\operatorname{span} M_1 \otimes \operatorname{span} M_2 = \operatorname{span}\{u \otimes v : u \in M_1, v \in M_2\}$ *holds.*

Proof. This follows from the definition of $H_1 \otimes H_2$. For $a_j, b_k \in \mathbb{F}$, $u_j \in M_1$, and $v_k \in M_2$ elements of $\operatorname{span} M_1 \otimes \operatorname{span} M_2$ are of the form $\sum_j a_j u_j \otimes \sum_k b_k v_k$ which denote the same conjugacy class as $\sum_{j,k} a_j b_k (u_j \otimes v_k)$, which is precisely the form of elements in $\operatorname{span}\{u \otimes v : u \in M_1, v \in M_2\}$. □

Now, we want to prove some results about the orthonormal basis of the tensor product space.

Theorem 15. *Let H_1 and H_2 be Hilbert spaces. If M_1 and M_2 are total subsets of H_1 and H_2 respectively, then the set $\{u \otimes v : u \in M_1, v \in M_2\}$ is total in $H_1 \hat{\otimes} H_2$.*

Proof. Let $\sum_{j=1}^n u_j \otimes v_j \in H_1 \otimes H_2$, and $\epsilon > 0$ be given. As spans of M_1 and M_2 are dense in H_1 and H_2 respectively, for every $j = 1, \ldots, n$ there exist elements $u_j' \in \operatorname{span} M_1$ and $v_j' \in \operatorname{span} M_2$ such that $\left\|u_j - u_j'\right\| \left\|g_j\right\| < \epsilon/2n$ and $\left\|v_j - v_j'\right\| \left\|f_j\right\| < \epsilon/2n$. We get

$$\begin{aligned}\left\|u_j \otimes v_j - u_j' \otimes v_j'\right\| &= \left\|(u_j - u_j') \otimes v_j + u_j' \otimes (v_j - v_j')\right\| \\ &\leq \left\|(u_j - u_j') \otimes v_j\right\| + \left\|u_j' \otimes (v_j - v_j')\right\| \\ &< \epsilon/n,\end{aligned}$$

and therefore

$$\left\|\sum_{j=1}^n u_j \otimes v_j - \sum_{j=1}^n u_j' \otimes v_j'\right\| < \epsilon. \tag{42}$$

As $\sum_j u_j' \times v_j' \in \operatorname{span} M_1 \otimes \operatorname{span} M_2 = \operatorname{span}\{u \otimes v : u \in M_1, v \in M_2\}$, we have shown that $\sum_j u_j \otimes f_j$ is a limit point of $\operatorname{span}\{u \otimes v : u \in M_1, v \in M_2\}$, and $\{u \otimes v : u \in M_1, v \in M_2\}$ is total in $H_1 \otimes H_2$. Since $H_1 \otimes H_2$ is dense in $H_1 \hat{\otimes} H_2$, the result follows. □

The next theorem gives an explicit orthonormal basis of $H_1 \hat{\otimes} H_2$.

Theorem 16. *Let H_1 and H_2 be Hilbert spaces. If $\{e_\alpha : \alpha \in A\}$ and $\{f_\beta : \beta \in B\}$ are orthonormal bases of H_1 and H_2 respectively, then $\{e_\alpha \otimes f_\beta : \alpha \in A, \beta \in B\}$ is an orthonormal basis of $H_1 \hat{\otimes} H_2$.*

Proof. By Theorem 15 the set $\{e_\alpha \otimes f_\beta : \alpha \in A, \beta \in B\}$ is total in $H_1 \hat{\otimes} H_2$. In addition, we have

$$\langle e_\alpha \otimes f_\beta, e_{\alpha'} \otimes f_{\beta'} \rangle = \langle e_\alpha, e_{\alpha'} \rangle \langle f_\beta, f_{\beta'} \rangle = \delta_{\alpha\alpha'} \delta_{\beta\beta'}, \tag{43}$$

i.e., $\{e_\alpha \otimes f_\beta : \alpha \in A, \beta \in B\}$ is an orthonormal system. Since it is also total, it is an orthonormal basis. □

2 Representations of $\mathfrak{sl}(2,\mathbb{C})$

We briefly review the notion of a Lie algebra.

2.1 Lie Algebras

Definition 18 (Lie algebra). *A finite-dimensional real or complex Lie algebra is a finite-dimensional real or complex vector space* $\mathfrak{g}$*, with a map* $[\cdot,\cdot]$ *from* $\mathfrak{g}\times\mathfrak{g}$ *into* $\mathfrak{g}$ *with the following properties*

1. $[\cdot,\cdot]$ *is bilinear,*
2. $[\cdot,\cdot]$ *is antisymmetric:* $[X,Y]=-[Y,X]$ *for all* $X,Y\in\mathfrak{g}$.
3. *Jacobi identity holds:* $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ *for all* $X,Y,Z\in\mathfrak{g}$.

The map $[\cdot,\cdot]$ is known as the *bracket* on $\mathfrak{g}$.

Definition 19 (Lie algebra homomorphism/isomorphism). *If* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are Lie algebras, then a linear map* $\phi:\mathfrak{g}\to\mathfrak{h}$ *is called a* Lie algebra homomorphism *if* $\phi([X,Y])=[\phi(X),\phi(Y)]$ *for all* $X,Y\in\mathfrak{g}$. *If, in addition,* ϕ *is one-to-one and onto, then* ϕ *is called a* Lie algebra isomorphism.

We can also define direct sum of two Lie algebras.

Definition 20 (Direct sum of Lie algebras). *If* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are Lie algebras, the direct sum of* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *is the vector space direct sum of* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$*, with bracket given by* $[(X_1,X_2),(Y_1,Y_2)]=([X_1,Y_1],[X_2,Y_2])$ *for* $X_1,Y_1\in\mathfrak{g}_1$ *and* $X_2,Y_2\in\mathfrak{g}_2$.

If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are subalgebras, we say that $\mathfrak{g}$ *decomposes as the Lie algebra direct sum* of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as vector spaces and $[X_1,X_2]=0$ for all $X_1\in\mathfrak{g}_1$ and $X_2\in\mathfrak{g}_2$.

Each matrix Lie group can be associated to a Lie algebra by the exponential map. Many questions about representations of groups can be transferred to a Lie algebra which, as it is a linear space, can be studied using linear algebra.

Definition 21 (Matrix exponential). *If* X *is an* $n\times n$ *matrix, we define the* exponential *of* X*, denoted* e^X *or* $\exp X$*, by the usual power series*

$$e^X=\sum_{m=0}^{\infty}\frac{X^m}{m!},\tag{44}$$

where $X^0:=I$.

Using properties of operator norm on the space of all finite-dimensional linear operators we have

$$\left\|e^X\right\|\leq\sum_{m=0}^{\infty}\left\|\frac{X^m}{m!}\right\|\leq\|I\|+\sum_{m=1}^{\infty}\frac{\|X\|^m}{m!}<\infty,\tag{45}$$

and the series defining e^X always converges. We shall need the following property of the matrix exponential.

Lemma 4. *For any* $X \in \mathsf{M}_n(\mathbb{C})$*, we have* $\det e^X = e^{\operatorname{Tr} X}$

Definition 22. *Let* G *be a matrix Lie group. The* Lie algebra of G *, denoted* $\mathfrak{g}$*, is the set of all matrices* X *such that* $e^{tX} \in G$ *for all real numbers* t*.*

It is easy to see that the set of matrices defined as above is a vector space. It can also be shown (using the Lie product formula) that $\mathfrak{g}$ as defined above is closed under the bracket defined by $[X, Y] = XY - YX$, so that $\mathfrak{g}$ is indeed an abstract Lie algebra as defined at the beginning of this section.

We now give several examples of Lie algebras associated to matrix Lie groups.

Proposition 19. *The Lie algebra of* $\mathrm{GL}(n, \mathbb{F})$ *is the space* $\mathsf{M}_n(\mathbb{F})$ *of all* $n \times n$ *matrices over* $\mathbb{F}$*.*

Lie algebra of $\mathrm{GL}(n, \mathbb{F})$ is denoted $\mathfrak{gl}(n, \mathbb{F})$.

Proof. If $X \in \mathsf{M}_n(\mathbb{F})$, then e^{tX} is invertible with $(e^{tX})^{-1} = e^{-tX}$, hence $X \in \mathfrak{gl}(n, \mathbb{F})$. With $\mathfrak{gl}(n, \mathbb{F}) \subset \mathsf{M}_n(\mathbb{F})$, we have $\mathfrak{gl}(n, \mathbb{F}) = \mathsf{M}_n(\mathbb{F})$. ☐

Proposition 20. *The Lie algebra of* $\mathrm{SL}(n, \mathbb{F})$ *consists of* $n \times n$ *matrices over* $\mathbb{F}$ *with trace zero.*

Lie algebra of $\mathrm{SL}(n, \mathbb{F})$ is denoted $\mathfrak{sl}(n, \mathbb{F})$.

Proof. If $X \in \mathsf{M}_n(\mathbb{F})$ has trace zero, by Lemma 4, $\det e^{tX} = 1$, showing that $X \in \mathfrak{sl}(n, \mathbb{F})$. Conversely, if $\det e^{tX} = e^{t \operatorname{Tr} X} = 1$ for all $t \in \mathbb{R}$, then

$$\operatorname{Tr} X = \frac{d}{dt} e^{t \operatorname{Tr} X} \bigg|_{t=0} = 0. \tag{46}$$

☐

Proposition 21. *The Lie algebra of* $\mathrm{U}(n)$ *consists of all complex matrices satisfying* $X^* = -X$*.*

Lie algebra of $\mathrm{U}(n)$ is denoted $\mathfrak{u}(n)$.

Proof. A matrix is unitary if and only if $U^* = U^{-1}$. Thus e^{tX} is unitary if and only if $(e^{tX})^* = e^{-tX}$, which holds if and only if $X^* = -X$. ☐

Combining the two results above, we can show that

Proposition 22. *The Lie algebra of* $\mathrm{SU}(n)$ *consists of all complex matrices satisfying* $X^* = -X$ *and* $\operatorname{Tr} X = 0$*.*

Lie algebra of $\mathrm{SU}(n)$ is denoted $\mathfrak{su}(n)$.

If $\mathfrak{g}$ is a finite-dimensional Lie algebra, there exists a finite basis $X_1, \ldots, X_n$ for $\mathfrak{g}$ (as a vector space). If we know the commutator brackets of basis elements with each other, antisymmetry and bilinearity will allow us to compute any other bracket.

2.2 Lie Algebra Representations

We now define representation of a Lie algebra.

Definition 23 (Representation of a Lie algebra). *Let $\mathfrak{g}$ be a Lie algebra, V be a real or complex vector space. A representation of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$.*

Like in the case of groups, the representation is a linear action of the Lie algebra on the vector space V.

The following theorem relates Lie group and Lie algebra homomorphisms. Note that, the theorem actually tells us how to relate Lie group and Lie algebra representations.

Theorem 17. *Let G and G be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Suppose that $\Phi : G \to H$ is a Lie group homomorphism, then there exists a unique real-linear map $\phi : \mathfrak{g} \to f\,h$ such that*

$$\Phi(e^X) = e^{\phi(X)} \tag{47}$$

for all $X \in \mathfrak{g}$. The map ϕ has the following additional properties:

1. $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$, *for all $X \in \mathfrak{g}$, $A \in G$.*
2. $\phi([X,Y]) = [\phi(X), \phi(Y)]$, *for all $X, Y \in \mathfrak{g}$.*
3. $\phi(X) = \frac{d}{dt}\Phi(e^{tX})\Big|_{t=0}$ *for all $X \in \mathfrak{g}$.*

With $H = \mathrm{GL}(V)$ and $\mathfrak{h} = \mathfrak{gl}(V)$, the above theorem tells us that given a Lie group representation $\Pi : G \to \mathrm{GL}(V)$, an associated representation of the Lie algebra $\mathfrak{g}$ on V is uniquely defined by $\pi(X) = \frac{d}{dt}\Pi(e^{tX})\Big|_{t=0}$ for every $X \in \mathfrak{g}$.

Definition 24 (Complexification). *If V is a finite dimensional real vector space, then the complexification of V, denoted $V_{\mathbb{C}}$, is the space of formal linear combinations $v_1 + iv_2$ with $v_1, v_2 \in V$. $V_{\mathbb{C}}$ becomes a real vector space in the obvious way and becomes a complex vector space if we define $i(v_1 + iv_2) = -v_2 + iv_1$.*

The following result states that the complexification of a real Lie algebra becomes a complex Lie algebra.

Proposition 23. *Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ be its complexification. Then the bracket operation on $\mathfrak{g}$ has a unique extension to $\mathfrak{g}_{\mathbb{C}}$ that makes $\mathfrak{g}_{\mathbb{C}}$ into a complex Lie algebra.*

The complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is called the complexification of the real Lie algebra $\mathfrak{g}$. Furthermore, if $\mathfrak{g} \subset \mathsf{M}_n(\mathbb{C})$, then the abstract complexification $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to the set of matrices in $\mathsf{M}_n(\mathbb{C})$ that can be written $X + iY$ for $X, Y \in \mathfrak{g}$. For example let $\mathfrak{g} = \mathfrak{u}(n)$. If $X^* = -X$, then $(iX)^* = iX$. Thus X and X^* cannot both be in $\mathfrak{u}(n)$. Furthermore, every X in $\mathsf{M}_n(\mathbb{C})$ can be expressed as $X = X_1 + iX_2$, where $X_1 = (X - X^*)/2$ and $X_2 = (X + X^*)/2i$ are both in $\mathfrak{u}(n)$. This shows that $\mathfrak{u}(n)_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$. For $\mathfrak{su}(n)_{\mathbb{C}}$, we have to take into account the extra requirement that $\mathrm{Tr}\,X = 0$ for every $X \in \mathfrak{su}(n)$. By this, we have $\mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n, \mathbb{C})$.

The following result tells us how representations of a Lie algebra are related to its complexification.

Proposition 24. *Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Every finite dimensional representation π of $\mathfrak{g}$ has a unique extension to a complex-linear representation of $\mathfrak{g}_{\mathbb{C}}$ also denoted π. Furthermore, π is irreducible as a representation of $\mathfrak{g}_{\mathbb{C}}$ if and only if it is irreducible as a representation of $\mathfrak{g}$.*

Finally, we state a result that gives a partial converse to Theorem 17.

Theorem 18. *Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $f\,h$ respectively, and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected, there exists a unique Lie group homomorphism $\Phi : G \to H$ such that $\Phi(e^X) = e^{\phi(X)}$ for all $X \in \mathfrak{g}$.*

With $H = \mathrm{GL}(V)$ and $\mathfrak{h} = \mathfrak{gl}(V)$, the above theorem (along with Theorem 17) tells us that there is a one-to-one correspondence between Lie group and Lie algebra representations in case of a simply connected group.

Definition 25. *Let $\mathfrak{g}$ be a Lie algebra, and π_1 and π_2 be representations of $\mathfrak{g}$ acting on V_1 and V_2. We define the* direct sum *of π_1 and π_2 acting on $V_1 \oplus V_2$ by*

$$[\pi_1 \oplus \pi_2(X)](v_1, v_2) = (\pi_1(X)v_1, \pi_2(X)v_2) \tag{48}$$

for all $X \in \mathfrak{g}$.

2.3 Tensor Product of $\mathfrak{sl}(2,\mathbb{C})$ Representations

As we have seen already, $\mathfrak{sl}(2,\mathbb{C})$ is the complexification of $\mathfrak{su}(2)$; irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ give unique irreducible representations of $\mathfrak{su}(2)$. Moreover, since SU(2) is simply connected, each Lie algebra representation gives rise to a unique group representation.

We recall that $\mathfrak{sl}(2,\mathbb{C})$ is the set of all traceless, complex matrices and choose the following basis for it:

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{49}$$

which have the commutation relations

$$[H,X] = 2X, \quad [H,Y] = -2Y, \quad [X,Y] = H. \tag{50}$$

If V is a finite-dimensional complex vector space and A, B and C are operators on V satisfying $[A,B] = 2B$, $[A,C] = -2C$, and $[B,C] = A$, then because of antisymmetry and bilinearity of brackets, the unique linear map $\pi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$ given by

$$\pi(H) = A, \quad \pi(X) = B, \quad \pi(Y) = C \tag{51}$$

defines a representation of $\mathfrak{sl}(2,\mathbb{C})$

To classify all representations of $\mathfrak{sl}(2,\mathbb{C})$, we need the following result.

Lemma 5. *Let u be an eigenvector of $\pi(H)$ with eigenvalue $\alpha \in \mathbb{C}$. Then we have*

$$\pi(H)\pi(X)u = (\alpha+2)\pi(X)u; \tag{52}$$

thus, either $\pi(X)u = 0$ or $\pi(X)u$ is an eigenvector of $\pi(H)$ with eigenvalue $\alpha+2$. Similarly,

$$\pi(H)\pi(Y)u = (\alpha-2)\pi(Y)u, \tag{53}$$

so that either $\pi(Y)u = 0$ or $\pi(Y)u$ is an eigenvector of $\pi(H)$ with eigenvalue $\alpha-2$.

Proof. We have to use $[\pi(H), \pi(X)] = \pi([H,X]) = 2\pi(X)$:

$$\begin{aligned}\pi(H)\pi(X)u &= \pi(X)\pi(H)u + [\pi(H), \pi(X)]u \\ &= \pi(X)(\alpha u) + 2\pi(X)u \\ &= (\alpha+2)\pi(X)u.\end{aligned}$$

Since, $[\pi(H), \pi(Y)] = \pi([H,Y]) = -2\pi(Y)$, we similarly have

$$\begin{aligned}\pi(H)\pi(Y)u &= \pi(Y)\pi(H)u + [\pi(H), \pi(Y)]u \\ &= \pi(Y)(\alpha u) - 2\pi(Y)u \\ &= (\alpha-2)\pi(Y)u.\end{aligned}$$

□

Theorem 19. *For each integer $m \geq 0$, there is an irreducible complex representation of $\mathfrak{sl}(2,\mathbb{C})$ with dimension $m+1$. Any two irreducible complex representations of $\mathfrak{sl}(2,\mathbb{C})$ with the same dimension are isomorphic.*

Proof. Let π be an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ acting on a finite dimensional complex vector space V. Since, we are working over $\mathbb{C}$, the operator $\pi(H)$ must have at least one eigenvector. Let $\pi(H)u = \alpha u$, for some non-zero $u \in V$ and $\alpha \in \mathbb{C}$. Repeatedly applying Lemma 5 yields

$$\pi(H)\pi(X)^k u = (\alpha+2k)\pi(X)^k u. \tag{54}$$

Since, the underlying space is finite dimensional, we can repeat the process only finitely many times before we encounter $\pi(X)^k u = 0$. Let $N \geq 0$ be such that $\pi(X)^N u \neq 0$ but $\pi(X)^{N+1} u = 0$.

Set $u_0 = \pi(X)^N u$ and $\lambda = \alpha + 2N$, so that $\pi(H)u_0 = \lambda u_0$ and $\pi(X)u_0 = 0$. We now want to lower the eigenvalue by repeated application of $\pi(Y)$; define $u_k = \pi(Y)^k u_0$ for $k \geq 0$. By Lemma 5, we have

$$\pi(H)u_k = \pi(H)\pi(Y)^k u_0 = (\lambda-2k)\pi(Y)^k u_0 = (\lambda-2k)u_k. \tag{55}$$

By induction over k, we can prove that

$$\pi(X)u_k = k[\lambda-(k-1)]u_{k-1} \quad (k \geq 1). \tag{56}$$

It is easy to see for $k=1$:

$$\pi(X)u_1 = \pi(X)\pi(Y)u_0 = \pi(Y)\pi(X)u_0 + \pi(H)u_0 = \lambda u_0. \tag{57}$$

Assuming the result for k, we have

$$\begin{aligned}\pi(X)u_{k+1} &= \pi(X)\pi(Y)u_k = [\pi(Y)\pi(X)+\pi(H)]u_k \\ &= k[\lambda-(k-1)]\pi(Y)u_{k-1} + (\lambda-2k)u_k \\ &= (k+1)(\lambda-k)u_k.\end{aligned}$$

Finally, since $\pi(H)$ can have only finitely many eigenvalues, the u_ks cannot all be non-zero. Hence, there must be a non-negative integer m such that $u_k = \pi(Y)^k u_0 \neq 0$ for all $k \leq m$, but $u_{m+1} = \pi(Y)^{m+1}u_0 = 0$.

Since $u_{m+1} = 0$, we have $\pi(X)u_{m+1} = 0$ and so by (56),

$$0 = \pi(X)u_{m+1} = (m+1)(\lambda - m)u_m. \tag{58}$$

As $m+1$ and u_m are non-zero, we must have $\lambda = m$.

Thus, for every irreducible representation (π, V), there exists an integer $m \geq 0$ and non-zero vectors $u_0, \ldots, u_m$ such that

$$\begin{aligned}\pi(H)u_k &= (m-2k)u_k \\ \pi(Y)u_k &= \begin{cases} u_{k+1} & \text{if } k < m \\ 0 & \text{if } k = m \end{cases} \\ \pi(X)u_k &= \begin{cases} k(m-(k-1))u_{k-1} & \text{if } k > 0 \\ 0 & \text{if } k = 0 \end{cases}\end{aligned} \tag{59}$$

Since the vectors $u_0, \ldots, u_m$ are eigenvectors for different eigenvalues, they must be linearly independent. The $(m+1)$-dimensional span of $u_0, \ldots, u_m$ is explicitly invariant under $\pi(H)$, $\pi(X)$ and $\pi(Y)$, and hence it is invariant under all of $\mathfrak{sl}(2, \mathbb{C})$. Since π is irreducible by hypothesis, the entire space V must be spanned by $u_0, \ldots, u_m$; in particular, $\dim V = m+1$. □

For every non-negative integer m, we shall denote the associate irreducible representation as π_m, which acts on the $m+1$-dimensional space V_m.

Since the element $H = \text{diag}(1, -1)$ of $\mathfrak{sl}(2, \mathbb{C})$ is in $i\mathfrak{su}(2)$, H is self-adjoint and so is $\pi(H)$. Hence, the eigenvectors of $\pi_m(H)$ with distinct eigenvalues are orthogonal.

Definition 26. *Let $\mathfrak{g}$ be a Lie algebra and let π_1 and π_2 be representations of $\mathfrak{g}$ acting on spaces V_1 and V_2 respectively. Then the* tensor product *of π_1 and π_2, denoted $\pi_1 \otimes \pi_2$, is a representation of $\mathfrak{g}$ acting on $V_1 \otimes V_2$ given by*

$$(\pi_1 \otimes \pi_2)(X) = \pi_1(X) \otimes I + I \otimes \pi_2(X) \tag{60}$$

for all $X \in \mathfrak{g}$.

If π_1 and π_2 are irreducible representations of $\mathfrak{g}$, then $\pi_1 \otimes \pi_2$ is typically not irreducible when viewed as a representation of $\mathfrak{g}$. In this final result, we shall attempt to decompose tensor product representations of $\mathfrak{sl}(2,\mathbb{C})$ as a direct sum of irreducible representations.

Theorem 20. *Let m and n be non-negative integers with $m \geq n$, and (π_m, V_m) and (π_n, V_n) be irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ as seen in Theorem 19. If we consider $\pi_m \otimes \pi_n$ as a representation of $\mathfrak{sl}(2,\mathbb{C})$, then*

$$\pi_m \otimes \pi_n \cong \pi_{m+n} \oplus \pi_{m+n-2} \oplus \ldots \oplus \pi_{m-n+2} \oplus \pi_{m-n}, \tag{61}$$

where $\cong$ denotes an isomorphism of $\mathfrak{sl}(2,\mathbb{C})$ representations.

Proof. We know that the eigenvectors of $\pi(H)$ form a basis for the irreducible representation space. Let us choose a basis $u_m, u_{m-2}, \ldots, u_{-m}$ for V_m and $v_n, v_{n-2}, \ldots, v_{-n}$ for V_n, with $\pi_m(H)u_j = j u_j$ and $\pi_n(H)v_k = k v_k$. By Theorem 16, $u_j \otimes v_k$ forms a basis for $V_m \otimes V_n$ and

$$[\pi_m(H) \otimes I + I \otimes \pi_n(H)]u_j \otimes v_k = (j+k)u_j \otimes v_k. \tag{62}$$

Hence, each basis element is an eigenvector for $\pi_m \otimes \pi_n(H)$. Thus, the eigenvalues of $\pi_m \otimes \pi_n(H)$ range from $m+n$ to $-(m+n)$ in increments of 2.

The eigenspace corresponding to eigenvalue $m+n$ is one dimensional, spanned by $u_m \otimes v_n$. If $n > 0$, the eigenspace corresponding to eigenvalue $m+n-2$ has dimension 2, and is spanned by $u_{m-2} \otimes v_n$ and $u_m \otimes v_{n-2}$. Each time the eigenvalue of $\pi_1 \otimes \pi_2(H)$ is decreased by 2, the dimension of the eigenspace increases by 1 until we reach the eigenvalue $m-n$, whose eigenspace is spanned by

$$u_{m-2n} \otimes v_n, u_{m-2n+2} \otimes v_{n-2}, \ldots, u_m \otimes v_{-n}. \tag{63}$$

This space has dimension $n+1$. Further decreasing the eigenvalue in increments of 2, leads to the dimension of representation remaining constant until we reach $n-m$, after which dimensions begin decreasing by 1 until we reach the lowest eigenvalue $-m-n$, for which the dimension is 1 and the eigenspace is spanned by $u_{-m} \otimes v_{-n}$.

Consider now, the vector $u_m \otimes v_n$ which is annihilated by $\pi_m \otimes \pi_n(X)$ and is an eigenvector for H with eigenvalue $m+n$. Applying $\pi_m \otimes \pi_n(Y)$ to $u_m \otimes v_n$ repeatedly gives a chain of eigenvectors of $\pi_m \otimes \pi_n(H)$ with eigenvalues decreasing by 2 until they reach $-m-n$. By the argument used in the proof of Theorem 19, the span W of these vectors is invariant under $\mathfrak{sl}(2,\mathbb{C})$, irreducible, and isomorphic to V_{m+n}.

As the orthogonal complement of an invariant subspace is also invariant, $W^{\perp}$ is also invariant. Since W contains exactly one eigenvector corresponding to each eigenvalue, dimension of each eigenspace in $W^{\perp}$ will be lowered by 1. In particular, $m+n$ will have no eigenvectors in $W^{\perp}$ and the next largest eigenvalue is $m+n-2$ which has multiplicity 1 (unless $n=0$). Thus, if we start with an eigenvector for $\pi_m \otimes \pi_n(H)$ in $W^{\perp}$ with eigenvalue $m+n-2$, this will be annihilated by $\pi_m \otimes \pi_n(X)$ and, on repeated application of $\pi_m \otimes \pi_n(Y)$ generate an irreducible, invariant subspace isomorphic to V_{m+n-2}.

We continue this process; choosing an eigenvector for the highest remaining eigenvalue in the orthogonal complement of the sum of all invariant subspaces that have been obtained in previous stages. Each such step reduces the dimension of each eigenspace by 1, and reduces the largest remaining eigenvalue by 2. This process continues till there is nothing left, which occurs after V_{m-n}. □

For example, the four dimensional representation $\pi_1 \otimes \pi_1$ decomposes as $\pi_2 \oplus \pi_0$, into a three dimensional and a one dimensional representation respectively.

In quantum mechanics, this computation is known as the *addition of angular momenta*. In the above example, the tensor product $\pi_1 \otimes \pi_1$ stands in for the total angular momentum of two "spin-1/2" particles. The decomposition $\pi_2 \oplus \pi_0$ tells us that the result is a three dimensional (triplet) invariant subspace with "spin-1", and a one dimensional (singlet) invariant subspace with "spin-0".

References

1. Joachim Weidmann. *Linear Operators in Hilbert Spaces* (1980). Springer-Verlag New York Inc.
2. Brian Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction* (2014). Second Edition. Springer International Publishing.