Group C*-algebras of Locally Compact Groups

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Definition (Topological group)

A *topological group* is a group, G, equipped with a topology such that the group operations of multiplication

$$(x, y) \mapsto xy, \quad G \times G \to G, \tag{1}$$

and inversion

$$x \mapsto x^{-1}, \quad G \to G, \tag{2}$$

are continuous.

Example (Examples of topological groups)

1. Finite groups with discrete topology
2. $\mathbb{R}^n$ with vector addition and usual topology
3. Matrix groups $GL(n, \mathbb{R}), O(n), GL(n, \mathbb{C}), U(n)$ with subspace topology inherited from $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$.

Locally compact group

Definition (Locally compact group)

A *locally compact group* is a topological group that is locally compact.

Definition (Haar measure)

Let G be a locally compact group. A *left Haar measure* on G is a nonzero Radon measure μ on G that satisfies $\mu(xE) = \mu(E)$ for every Borel set $E \subseteq G$ and every $x \in G$.

Theorem (Haar's theorem)

Every locally compact group possesses a left Haar measure. Moreover, it is unique upto multiplication by a positive number.

Example (Examples of Haar measure)

1. Finite groups with the normalized counting measure
2. $\mathbb{R}^n$ with the Lebesgue measure
3. $GL(n, \mathbb{R})$ with $d\mu(x) = |\det x|^{-n} \prod_{i,j=1}^{n} dx_{ij}$

Modular function

If μ is a Haar measure on the group G, then μ_x defined by $\mu_x(E) = \mu(Ex)$ is also a left invariant measure on G. By uniqueness of Haar measure these should be related by a positive number $\Delta(x)$ in the following way: $\mu_x = \Delta(x)\mu$.

$\Delta : G \to (0, \infty)$ is called the *modular function* of G.

Proposition

For any $f \in L^1(G, \mu)$, $\int_G f(xy)\,dx = \Delta(y^{-1}) \int_G f(x)\,dx$.

Proposition

The modular function $\Delta : G \to (\mathbb{R}_+)^\times$ *is a continuous group homomorphism.*

Proposition

For any $f \in L^1(G, \mu)$, $\int_G f(x^{-1})\,dx = \int_G f(x)\Delta(x^{-1})\,dx$.

Denote the space of bounded and unitary operators on a Hilbert space, H, by $B(H)$ and $U(H)$ respectively.

Definition (Unitary representation of group)

Let G be a locally compact group. A *unitary representation of* G on the Hilbert space H is a group homomorphism $\pi : G \to U(H)$, which is continuous with respect to the strong operator topology.

Definition (Invariant subspace)

Let (π, H) be a representation of G. A closed subspace $K \subseteq H$ is said to be *invariant* if $\pi(x)u \in K$ for all $x \in K$.

Definition (Irreducible representation)

A representation (π, H), is said to be *irreducible* if the only invariant subspaces of H are $\{0\}$ and H.

Translations of functions

Definition (Left and right translations)
Let $f : G \to \mathbb{C}$ be a function on G. The *left translation* of f by $y \in G$ is defined to be the function $L_y f : G \to \mathbb{C}$ given by $L_y f(x) = f(y^{-1}x)$.

Similarly, the *right translation* of f by $y \in G$ is given by $R_y f(x) = f(xy)$.

Left regular representation

Consider G acting on the Hilbert space $L^2(G) := L^2(G, \mu)$ by left translations.

$$\pi_L : G \to U(L^2(G)), \quad \pi_L(x)f = L_x f \text{ for } f \in L^2(G) \tag{3}$$

It is clear that π_L is a homomorphism,

$$\pi_L(xy)f(z) = f(y^{-1}x^{-1}z) = L_y f(x^{-1}z) = L_x L_y f(z) = \pi_L(x)\pi_L(y)f(z). \tag{4}$$

To see that $\pi_L(x)$ is unitary and bounded note that for any $f \in L^2(G)$

$$\left\|\pi_L(x)f\right\|_2^2 = \left\|L_x f\right\|_2^2 = \int_G \overline{L_x f(y)} L_x f(y)\, dy = \int_G \overline{f(x^{-1}y)} f(x^{-1}y)\, dy \tag{5}$$

$$= \int_G \overline{f(y)} f(y)\, dy = \|f\|_2^2. \tag{6}$$

$L^1(G) := L^1(G, \mu)$ is a Banach space.

Let $f, g \in L^1(G)$. With the convolution product, $(f * g)(x) = \int_G f(y)g(y^{-1}x)\,dy$, it becomes a Banach algebra.

Indeed, using Tonelli's theorem to exchange order of integrals,

$$\|f * g\|_1 = \int_G \left| \int_G f(y)g(y^{-1}x)\,dy \right| dx \leq \iint_{G\times G} |f(y)g(y^{-1}x)|\,dy\,dx \tag{7}$$

$$= \int_G |f(y)| \left(\int_G |g(y^{-1}x)|\,dx \right) dy \tag{8}$$

$$= \int_G |f(y)|\, \|g\|_1\,dy = \|f\|_1\, \|g\|_1 < \infty. \tag{9}$$

$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$ defines an involution on $L^1(G)$:

$$(f^*)^*(x) = \Delta(x^{-1})\overline{f^*(x^{-1})} = \Delta(x^{-1})\Delta(x)f(x) = f(x), \tag{10}$$

$$\begin{aligned}(f + \lambda g)^*(x) &= \Delta(x^{-1})(\overline{f + \lambda g})(x^{-1}) \\ &= \Delta(x^{-1})(\overline{f(x^{-1})} + \overline{\lambda g(x^{-1})}) = f^*(x) + \bar{\lambda} g^*(x),\end{aligned} \tag{11}$$

$$\begin{aligned}(f * g)^*(x) &= \Delta(x^{-1})\overline{f * g(x^{-1})} = \Delta(x^{-1}) \int_G \overline{f(y)g(y^{-1}x^{-1})}\, dy \\ &= \Delta(x^{-1}) \int_G \overline{f(x^{-1}y)g(y^{-1})}\, dy \\ &= \Delta(x^{-1}) \int_G \Delta(y)\overline{f(x^{-1}y)}\Delta(y^{-1})\overline{g(y^{-1})}\, dy \\ &= \int_G \Delta(y^{-1})\overline{g(y^{-1})}\Delta((y^{-1}x)^{-1})\overline{f((y^{-1}x)^{-1})}\, dy \\ &= \int_G g^*(y)f^*(y^{-1}x)\, dy = g^* * f^*(x),\end{aligned} \tag{12}$$

$$\|f^*\|_1 = \int_G |\Delta(x^{-1})\overline{f(x^{-1})}|\, dx = \int_G |\overline{f(x)}|\, dx = \|f\|_1 . \tag{13}$$

With the convolution product and involution as defined above, $L^1(G)$ has the structure of a Banach $*$-algebra.

Definition (Group algebra)

Let G be a locally compact group. The Banach $*$-algebra, $L^1(G)$ equipped with the convolution product and involution as defined above, is called the *group algebra* of G.

Definition ($*$-algebra representation)

Let A be a $*$-algebra. A *$*$-representation* of A on a Hilbert space H is a $*$-algebra homomorphism $\pi : A \to B(H)$.

Invariant subspaces and irreducibility are defined exactly as in the case of groups.

Definition (Nondegenerate representation)

A $*$-representation (π, H), of A is said to be *nondegenerate* if for every $u \in H$, there exists $f \in A$ such that $\pi(f)u \neq 0$.

Group algebra representation from group representation

Given a unitary representation (π, H) of G, define a representation of $L^1(G)$ on the same Hilbert space H by,

$$\tilde{\pi}(f) = \int_G f(x)\pi(x)\,dx, \quad \text{for } f \in L^1(G), \tag{14}$$

where the integral is interpreted in the weak sense, i.e., for all $u, v \in H$,

$$\langle u|\tilde{\pi}(f)v\rangle = \int_G f(x)\langle u|\pi(x)v\rangle\,dx. \tag{15}$$

$\tilde{\pi}(f)$ is bounded. Indeed, $|\langle u|\tilde{\pi}(f)v\rangle| \leq \|f\|_1 \|u\| \|v\|$, therefore $\|\tilde{\pi}(f)\| \leq \|f\|_1$.

The correspondence $f \mapsto \tilde{\pi}(f)$ is linear.

Moreover,

$$\tilde{\pi}(f * g) = \int_G (f * g)(x)\pi(x)\,dx = \iint_{G\times G} f(y)g(y^{-1}x)\pi(x)\,dy\,dx \tag{16}$$

$$= \iint_{G\times G} f(y)g(z)\pi(yz)\,dy\,dz = \iint_{G\times G} f(y)g(z)\pi(y)\pi(z)\,dy\,dz \tag{17}$$

$$= \left(\int_G f(y)\pi(y)\,dy\right)\left(\int_G g(z)\pi(z)\,dz\right) \tag{18}$$

$$= \tilde{\pi}(f)\tilde{\pi}(g), \tag{19}$$

and

$$\tilde{\pi}(f^*) = \int_G \overline{f(x^{-1})}\Delta(x^{-1})\pi(x)\,dx = \int_G \overline{f(x)}\pi(x^{-1})\,dx \tag{20}$$

$$= \int_G \overline{f(x)}\pi(x)^*\,dx = \int_G [f(x)\pi(x)]^*\,dx = \tilde{\pi}(f)^*. \tag{21}$$

Therefore, $\tilde{\pi}$ is a $*$-representation of $L^1(G)$ on H.

Lemma

Let (π, H) be a unitary representation of a locally compact group, G. Then $\tilde{\pi}$ as defined above is a $*$-representation of the group algebra, $L^1(G)$, on H.

Lemma

The representation $\tilde{\pi}$ is nondegenerate.

Proof.

Let $u \in H$. By continuity of π, there exists a compact neighborhood, V, of 1_G such that $\|\pi(x)u - u\| < \|u\|$ for $x \in V$. With $f = \mu(V)^{-1}\chi_V$,

$$\|\tilde{\pi}(f)u - u\| = \left\|\left(\mu(V)^{-1} \int_V \pi(x)\,dx\right)u - u\right\| \tag{22}$$

$$= \mu(V)^{-1} \left\|\int_V (\pi(x)u - u)\,dx\right\| \tag{23}$$

$$\leq \mu(V)^{-1} \int_V \|\pi(x)u - u\|\;dx < \|u\|, \tag{24}$$

therefore $\|\tilde{\pi}(f)u\| \neq 0$. □

Putting the two results together,

Theorem

Let (π, H) be a unitary representation of G. The map $f \mapsto \tilde{\pi}(f)$ is a nondegenerate $*$-representation of the group algebra $L^1(G)$ on H.

Left regular representation

Recall the left regular representation of G on $L^2(G)$

$$\pi_L : G \to U(L^2(G)), \quad \pi_L(x)f = L_x f. \tag{25}$$

The $*$-representation induced by π_L on the group algebra acts by

$$\tilde{\pi}_L(f) = \int_G f(x)\pi_L(x)\,dx = \int_G f(x)L_x\,dx, \tag{26}$$

so that

$$\tilde{\pi}_L(f)g = \int_G f(x)L_x g\,dx = f * g, \tag{27}$$

for $f \in L^1(G)$ and $g \in L^2(G)$.

(Foreshadowing) There is at least one bounded representation of $L^1(G)$.

Group representation from group algebra representation

The converse

Theorem

If $(\tilde{\pi}, H)$ is a nondegenerate $*$-representation of $L^1(G)$, then it arises from a unique unitary representation, π, of G on H.

is also true.

Definition (Group C*-algebra)

Let G be a locally compact group. The *group C*-algebra*, $C^*(G)$, is defined as the universal enveloping C*-algebra of the group algebra $L^1(G)$.

To see that $C^*(G)$ is well defined, note that

1. $(\tilde{\pi}_L, L^2(G))$ is a bounded $*$-representation of $L^1(G)$
2. For any bounded representation ρ of $L^1(G)$, we have $\|\rho(f)\| \leq \|f\|_1 < \infty$, and therefore $\sup_{\text{bdd repns}} \|\rho(f)\| < \infty$ for every $f \in L^1(G)$.
3. $\ker \tilde{\pi}_L$ is trivial because $\tilde{\pi}_L(f) = 0 \implies \tilde{\pi}_L(f)g = 0$ for all $g \in L^2(G)$.
   Choose g to be an approximate identity $\{\psi_\alpha\}$ so that
   $\tilde{\pi}_L(f)\psi_\alpha = f * \psi_\alpha = 0$, therefore $f = 0$.
   In particular, $\cap_{\text{bdd repn}} \ker \rho$ is trivial.

This makes $L^1(G)$ into a normed $*$-algebra with respect to the *universal norm*,

$$\|f\|_u = \sup_{\text{bdd repn}} \|\rho(f)\| . \tag{28}$$

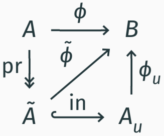
$C^*(G)$ is the norm completion of $L^1(G)$ with respect to the universal norm.

C*-property is satisfied in $L^1(G)$ with respect to $\|\cdot\|_u$ because operators in $B(H)$ satisfy the C*-property.

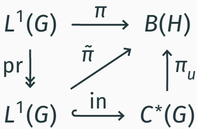
To establish a one-to-one correspondence between unitary representations of G and nondegenerate representations of $C^*(G)$, we use the universal property of C*-algebras.

Theorem (Universal property of universal enveloping C*-algebras)

Let A_u be the universal enveloping C-algebra of a $*$-algebra A, $\tilde{A} = A/\cap_\rho \ker\rho$, and let B be a C*-algebra. For any $*$-algebra homomorphism $\phi : A \to B$, there exists a unique $*$-algebra homomorphism $\phi_u : A_u \to B$ rendering the following diagram commutative*



Given a representation (π, H) of $L^1(G)$, we apply the above theorem with $A = L^1(G)$, $B = B(H)$, $\phi = \pi$, and $A_u = C^*(G)$, to get a unique representation of the group C*-algebra on H.



Moreover, π nondegenerate $\iff$ π_u nondegenerate.

Finally, as we have established

- A one-to-one correspondence between unitary representations of G and nondegenerate representations of $L^1(G)$, and
- A one-to-one correspondence between nondegenerate representations of $L^1(G)$ and $C^*(G)$,

$$\begin{array}{ccc} L^1(G) & \xrightarrow{\ \pi\ } & B(H) \\ & {}_{\text{in}}\searrow & \uparrow \pi_u \\ & & C^*(G) \end{array}$$

We have

Theorem

Let G be a locally compact group. Every unitary representation of G induces a nondegenerate representation of $C^*(G)$ on the same Hilbert space. Conversely, every nondegenerate representation of $C^*(G)$ is induced by a unique unitary representation of G.

Group representation from group algebra representation (proof)

In the following slides, we are going to construct a unique, unitary representation of G from a given nondegenerate representation of the group algebra $L^1(G)$.

Let $(\tilde{\pi}, H)$ be a representation of $L^1(G)$ arising from the representation (π, H) of G. For $x \in G$, $f \in L^1(G)$,

$$\pi(x)\tilde{\pi}(f) = \pi(x)\int_G f(y)\pi(y)\,dy = \int_G f(y)\pi(xy)\,dy \tag{29}$$

$$= \int_G f(x^{-1}z)\pi(z)\,dz = \int_G L_x f(z)\pi(z)\,dz \tag{30}$$

$$= \tilde{\pi}(L_x f). \tag{31}$$

Given a $*$-representation $(\tilde{\pi}, H)$ of $L^1(G)$, fix $x \in G$ and define $\pi_G(x) : K \to H$ on the subspace

$$K = \{\tilde{\pi}(f)u : f \in L^1(G), u \in H\}, \tag{32}$$

by $\pi_G(x)\tilde{\pi}(f)u = \tilde{\pi}(L_x f)u$.

Definition (Approximate identity)

Let A be a Banach algebra. A net, $\{x_\alpha\}$, in A is called an *approximate identity* if

1. $\|x_\alpha\| = 1$,
2. $\|x_\alpha a - a\| \to 0$ and $\|a x_\alpha - a\| \to 0$ for every $a \in A$.

Example (Approximate identity of $L^1(G)$)

Let $\{V_\alpha\}$ be a neighbourhood system for 1_G of symmetric compact sets. $\psi_\alpha = \mu(V_\alpha)^{-1}\chi_V$ is an approximate identity of $L^1(G)$.

Lemma

$\pi_G(x)$ is bounded on K and $\left\|\pi_G(x)\right\| \leq 1$.

Proof.

Note that as $\tilde{\pi}$ is a $*$-representation, $\|\tilde{\pi}\| \leq 1$.

Let $\{\psi_U\}$ be an approximate identity of $L^1(G)$. For any $f \in L^1(G)$, $L_x\psi_U * f = L_x(\psi_U * f) \xrightarrow{L^1} L_x f$.

$$\left\|\pi_G(x)\tilde{\pi}(f)u\right\| = \left\|\tilde{\pi}(L_x f)u\right\| = \lim_U \left\|\tilde{\pi}(L_x\psi_U * f)u\right\| \tag{33}$$

$$\leq \lim_U \left\|\tilde{\pi}(L_x\psi_U)\right\| \|\tilde{\pi}(f)u\| \tag{34}$$

$$\leq \lim_U \left\|L_x\psi_U\right\| \|\tilde{\pi}(f)u\| \leq \|\tilde{\pi}(f)u\| < \infty \tag{35}$$

In particular, $\left\|\pi_G(x)\right\| \leq 1$. □

Lemma

K is a dense subspace of H.

Proof.

Suppose there is a $v \in H$ such that $v \perp K$. Then, $\langle \tilde{\pi}(f)u|v\rangle = \langle u|\tilde{\pi}(f^*)v\rangle = 0$ for every $f \in L^1(G)$ and $u \in H$. Since $\tilde{\pi}$ is nondegenerate, $v = 0$.

$K^\perp = \{0\}$ therefore $\overline{K} = (K^\perp)^\perp = H$. □

As $\pi_G(x)$ is a bounded operator on a dense subspace, it has a unique norm preserving extension to H.

Lemma

$\pi_G : G \to B(H)$ is a group homomorphism.

Proof.

Let $x, y \in G$, $f \in L^1(G)$ and $u \in H$, then

$$\pi_G(xy)\tilde{\pi}(f)u = \tilde{\pi}(L_{xy}f)u = \tilde{\pi}(L_x L_y f)u \tag{36}$$

$$= \pi_G(x)\tilde{\pi}(L_y f)u \tag{37}$$

$$= \pi_G(x)\pi_G(y)\tilde{\pi}(f)u, \tag{38}$$

therefore $\pi_G(xy) = \pi_G(x)\pi_G(y)$. □

Lemma

Let $x \in G$. $\pi_G(x)$ is unitary.

Proof.

$$\|u\| = \left\|\pi_G(x^{-1})\pi_G(x)u\right\| \leq \left\|\pi_G(x)u\right\| \leq \|u\|, \tag{39}$$

using $\left\|\pi_G(g)\right\| \leq 1$ for every $g \in G$. □

We have shown that (π_G, H) is a unitary representation of G.

The penultimate thing to do is show that $\tilde{\pi}_G = \tilde{\pi}$.

Lemma

$\tilde{\pi}_G = \tilde{\pi}$.

Proof.

We will use the fact that $\tilde{\pi} : L^1(G) \to B(H)$ is a bounded linear map and therefore commutes with the integral. Let $f, g \in L^1(G)$,

$$\begin{aligned}
\tilde{\pi}(f)\tilde{\pi}(g) &= \tilde{\pi}(f * g) && (40)\\
&= \tilde{\pi}\left(\int_G f(y)L_y g\,dy\right) = \int_G f(y)\tilde{\pi}(L_y g)\,dy && (41)\\
&= \int_G f(y)\pi_G(y)\tilde{\pi}(g)\,dy && (42)\\
&= \tilde{\pi}_G(f)\tilde{\pi}(g). && (43)
\end{aligned}$$

Since the choice of f was arbitrary, $\tilde{\pi}(f) = \tilde{\pi}_G(f)$ for all $f \in L^1(G)$. □

Lemma

Let τ and π be unitary representations of G on a Hilbert space H, and let $\tilde{\tau}$ and $\tilde{\pi}$ be corresponding $*$-representations of $L^1(G)$. If $\tilde{\tau}(f) = \tilde{\pi}(f)$ for all $f \in L^1(G)$, then $\tau(x) = \pi(x)$ for all $x \in G$.

Proof.

For all $u, v \in H$,

$$\langle u|\tilde{\tau}(f)v\rangle = \langle u|\tilde{\pi}(f)v\rangle \implies \int_G f(x)(\langle u|\tau(x)v\rangle - \langle u|\pi(x)v\rangle)\, dx = 0.$$

As $f \in L^1(G)$ is arbitrary, $\langle u|\tau(x)v\rangle = \langle u|\pi(x)v\rangle$ for every $x \in G$ and every $u, v \in H$. Therefore $\tau(x) = \pi(x)$ for every $x \in H$. □

Putting all of the results together, we have shown

Theorem

If $(\tilde{\pi}, H)$ is a nondegenerate $*$-representation of $L^1(G)$, then it arises from a unique unitary representation, π, of G on H.

With the converse

Theorem

Let (π, H) be a unitary representation of G. The map $f \mapsto \tilde{\pi}(f)$ is a nondegenerate $*$-representation of the group algebra $L^1(G)$ on H.

We have established a one-to-one correspondence between unitary representations of groups and nondegenerate representations of group algebras.

Thank you.