Analysis on the Euclidean Motion Group

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1 Isometries of $\mathbb{R}^n$

Definition 1 (isometry). *An* isometry *of $\mathbb{R}^n$ is a function $A : \mathbb{R}^n \to \mathbb{R}^n$ that preserves distances between points, i.e., for $x, y \in \mathbb{R}^n$ an isometry satisfies $||A(x) - A(y)|| = ||x - y||$ where*

$$||x|| = \sqrt{\sum_{j=1}^{n} x_j^2}. \tag{1}$$

We denote the collection of isometries by

$$I(n) = \left\{A : \mathbb{R}^n \to \mathbb{R}^n \;\middle|\; ||A(x) - A(y)|| = ||x - y|| \text{ for every } x, y \in \mathbb{R}^n\right\}.$$

An isometry is said to *fix the origin* if it satisfies $A(0) = 0$. It can be shown that isometries that keep the origin fixed preserve the dot product on $\mathbb{R}^n$.

Proposition 1. *A function $A : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry satisfying $A(0) = 0$ if and only if A preserves dot products: $\langle A(x), A(y)\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$.*

Proof. Let $A(0) = 0$. Since A fixes the origin, we have $||x|| = ||A(x) - A(0)|| = ||A(x)||$. Since $||x||^2 = \langle x, x\rangle$ we get

$$\begin{aligned}
&\langle A(x) - A(y), A(x) - A(y)\rangle = \langle x - y, x - y\rangle \\
\implies &\langle A(x), A(x)\rangle - 2\langle A(x), A(y)\rangle + \langle A(y), A(y)\rangle = \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle \\
\implies &\langle A(x), A(y)\rangle = \langle x, y\rangle
\end{aligned}$$

Conversely, assume that $\langle A(x), A(y)\rangle = \langle x, y\rangle$. We then have

$$\begin{aligned}
||A(x) - A(y)||^2 &= \langle A(x) - A(y), A(x) - A(y)\rangle \\
&= \langle A(x), A(x)\rangle - 2\langle A(x), A(y)\rangle + \langle A(y), A(y)\rangle \\
&= \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle \\
&= \langle x - y, x - y\rangle = ||x - y||^2. \qquad (2)
\end{aligned}$$

Hence, A is an isometry. Finally, setting $x = y = 0$ yields $||A(0)|| = 0$, and therefore $A(0) = 0$. □

Moreover, we can also show the following

Proposition 2. *A function $A : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry satisfying $A(0) = 0$ if and only if A is linear and orthogonal $AA^T = \mathbb{1}$.*

Proof. It suffices to show that the map A is linear as orthogonality follows because A preserves inner products:

$$\langle x, y\rangle = \langle A(x), A(y)\rangle = \langle A^T A(x), y\rangle = \langle x, AA^T(y)\rangle \tag{3}$$

for all $x, y \in \mathbb{R}^n$, and therefore $A^T A = AA^T = \mathbb{1}$. Let $\{e_j\}_{j=1,\dots,n}$ be the standard orthonormal basis for $\mathbb{R}^n$ such that $\langle e_j, e_k\rangle = \delta_{jk}$. Then $\{A(e_j)\}$ is also an orthonormal basis for $\mathbb{R}^n$ with $\langle A(e_j), A(e_k)\rangle = \delta_{jk}$. We shall first show that $A(cx) = cA(x)$. Let $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then $A(cx)$ can be expanded in the orthogonal basis $\{A(e_j)\}$

$$\begin{aligned} A(cx) &= \sum_{j=1}^{n} \langle A(cx), A(e_j)\rangle A(e_j) = \sum_{j=1}^{n} \langle cx, e_j\rangle A(e_j) \\ &= c\sum_{j=1}^{n} \langle x, e_j\rangle A(e_j) = c\sum_{j=1}^{n} \langle A(x), A(e_j)\rangle A(e_j) \\ &= cA(x). \end{aligned} \tag{4}$$

Similarly, for $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned} A(x+y) &= \sum_{j=1}^{n} \langle A(x+y), A(e_j)\rangle A(e_j) = \sum_{j=1}^{n} \langle x, e_j\rangle A(e_j) + \sum_{j=1}^{n} \langle y, e_j\rangle A(e_j) \\ &= \sum_{j=1}^{n} \langle A(x), A(e_j)\rangle A(e_j) + \sum_{j=1}^{n} \langle A(y), A(e_j)\rangle A(e_j) \\ &= A(x) + A(y). \end{aligned} \tag{5}$$

For the converse, we only note that, by definition, an orthogonal linear map preserves inner products and therefore by Proposition 1 it is an isometry which fixes the origin. □

The following result shows that any isometry of $\mathbb{R}^n$ can be written as a composition of a translation and an orthogonal map.

Theorem 1. *Every isometry of $\mathbb{R}^n$ can be written as $T \circ R$, where T is a translation and R is an orthogonal map.*

Proof. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry. For $x \in \mathbb{R}^n$ define a translation $T(x) := x + A(0)$ and an orthogonal map $R(x) := A(x) - A(0)$ so that we get $T \circ R(x) = T(A(x) - A(0)) = A(x)$. It follows that R as defined above is orthogonal because it is an isometry which fixes the origin, $A(0) = 0$.

Conversely, if T_w is a translation by a vector w and R is an orthogonal map so that

$A = T_w \circ R$, then for every $x, y \in \mathbb{R}$ we have

$$A(x) - A(y) = (R(x) + w) - (R(y) + w) = R(x) - R(y) \tag{6}$$

and therefore

$$||A(x) - A(y)|| = ||R(x) - R(y)|| = ||x - y||. \tag{7}$$

Hence, $T_w \circ R$ is an isometry. □

With these results, we can show that isometries of $\mathbb{R}^n$ are invertible and that the inverse of an isometry is also an isometry.

Proposition 3. *Isometries of $\mathbb{R}^n$ are invertible and the inverse of an isometry is also an isometry.*

Proof. Let $A \in I(n)$ be an isometry. By Theorem 1, $A = T \circ R$ where $T(x) = x + A(0)$ and R is an orthogonal map. As R is orthogonal, it is invertible with $R^{-1} = R^T$ and we define the inverse of A as

$$A^{-1}(x) = R^{-1}(x - A(0)). \tag{8}$$

To show that A^{-1} is an isometry, we note that

$$\left\| A^{-1}(x) - A^{-1}(y) \right\| = \left\| R^{-1}(x - A(0)) - R^{-1}(y - A(0)) \right\|, \tag{9}$$

for all $x, y \in \mathbb{R}^2$, and since $R^{-1} = R^T$ is an orthogonal map and therefore an isometry, we have

$$\left\| R^{-1}(x - A(0)) - R^{-1}(y - A(0)) \right\| = ||(x - A(0)) - (y - A(0))|| = ||x - y||.$$

Hence, if $A = T_w \circ R$ then $A^{-1} = T_{-R^{-1}w} \circ R^{-1}$. □

Hence, with the operation of function composition, the collection $I(n)$ becomes a group. For any two isometries $A, B \in I(n)$ we can write $A = T_{w_1} \circ R_1$ and $B = T_{w_2} \circ R_2$ where $R_j \in \mathrm{O}(n)$ and T_{w_j} are translations. We have the group composition for any $x \in \mathbb{R}^2$

$$A \circ B(x) = R_1 R_2(x) + w_1 + R_1(w_2) = T_{w_1 + R_1 w_2} \circ R_1 R_2. \tag{10}$$

And the inverse $A^{-1} = (T_w \circ R)^{-1} = R^{-1} \circ T_{-w} = T_{-R^{-1}w} \circ R^{-1}$, by Proposition 3.

Before moving ahead, we quickly note the following result.

Proposition 4. *The groups $\mathbb{R}^n$ and $\mathrm{O}(n)$ are subgroups of $I(n)$.*

1.1 Semidirect products

Definition 2 (semidirect product of groups). *Given a group K, a group N and an action Φ of K on N by automorphisms*

$$\Phi_k : N \to N \quad n \mapsto \Phi_k(n), \tag{11}$$

the semidirect product $N \rtimes K$ is the set of pairs $(n,k) \in N \times K$ with group composition law

$$(n_1,k_1)(n_2,k_2) = (n_1\Phi_{k_1}(n_2), k_1k_2). \tag{12}$$

Proposition 5. *Semidirect product of groups as defined above is indeed a group.*

Proof. Let $e_N \in N$ and $e_K \in K$ be the identities in N and K respectively. Then (e_N, e_K) is the identity for $N \rtimes K$

$$(e_N, e_K)(n,k) = (e_N\Phi_{e_K}(n), e_Kk) = (n,k) \tag{13}$$

and

$$(n,k)(e_N, e_K) = (n\Phi_k(e_N), ke_K) = (n,k). \tag{14}$$

Given the identity, we can compute the inverse with respect to the group composition law by requiring $(n,k)(n,k)^{-1} = (e_N, e_K)$. We can verify that the inverse is given by $(n,k)^{-1} = (\Phi_{k^{-1}}(n^{-1}), k^{-1})$:

$$\begin{aligned}(n,k)^{-1}(n,k) &= (\Phi_{k^{-1}}(n^{-1}), k^{-1})(n,k)\\ &= (\Phi_{k^{-1}}(n^{-1})\Phi_k(n), k^{-1}k)\\ &= (e_N, e_K).\end{aligned} \tag{15}$$

Finally, we verify that the group multiplication is associative. For $n_1, n_2, n_3 \in N$ and $k_1, k_2, k_3 \in K$

$$\begin{aligned}[(n_1,k_1)(n_2,k_2)](n_3,k_3) &= (n_1\Phi_{k_1}(n_2), k_1k_2)(n_3,k_3)\\ &= (n_1\Phi_{k_1}(n_2)\Phi_{k_1k_2}(n_3), k_1k_2k_3)\\ &= (n_1\Phi_{k_1}(n_2\Phi_{k_2}(n_3)), k_1k_2k_3)\\ &= (n_1,k_1)(n_2\Phi_{k_2}(n_3), k_2k_3)\\ &= (n_1,k_1)[(n_2,k_2)(n_3,k_3)].\end{aligned} \tag{16}$$

Hence, $N \rtimes K$ is indeed a group. □

Elements of $N \rtimes K$ of the kind (n, e_K) form a subgroup of $N \rtimes K$ isomorphic to N. Similarly, elements of the kind (e_N, k) form a subgroup isomorphic to K. In slight abuse of notation, when we shall say that N and K are subgroups of $N \rtimes K$ when we are actually referring to isomorphic copies of N and K inside $N \rtimes K$.

Proposition 6. *Let $N \rtimes K$ be a semidirect product of groups. Then N is a normal subgroup of $N \rtimes K$.*

Proof. We want to show that $gNg^{-1} = N$ for all $g = (n,k) \in N \rtimes K$. Let $(m, e_K) \in N \rtimes K$ for $m \in N$. We have

$$\begin{aligned}(n,k)(m,e_K)(n,k)^{-1} &= (n\Phi_k(m),k)(\Phi_{k^{-1}}(n^{-1}),k^{-1}) \\ &= (n\Phi_k(m)\Phi_k(\Phi_{k^{-1}}(n^{-1})),kk^{-1}) \\ &= (n\Phi_k(m)n^{-1},e_K) \in N. \end{aligned} \tag{17}$$

Thus, given any $m \in N$, we have for any $(n,k) \in N \rtimes K$, $n^{-1}mn \in N$ and $\Phi_{k^{-1}}(n^{-1}mn) \in N$ so that $(n,k)(\Phi_{k^{-1}}(n^{-1}mn),e_K)(n,k)^{-1} = (m,e_K)$ and therefore $(m,e_K) \in gNg^{-1}$ where $g = (n,k)$ and therefore $N \subset gNg^{-1}$. Conversely, any element of gNg^{-1} is of the form in (17) and hence $N = gNg^{-1}$ for all $g \in N \rtimes K$. □

The factor K in $N \rtimes K$ need not be normal. We also note that the direct product is a special case of semidirect product when Φ_k is the trivial automorphism for all $k \in K$.

We have an action Φ of $\mathrm{O}(n)$ on $\mathbb{R}^n$ by automorphisms: $\Phi_R : \mathbb{R}^n \to \mathbb{R}^n$ given naturally by $x \mapsto Rx$. We notice that the group composition law for a semidirect product $\mathbb{R}^n \rtimes \mathrm{O}(n)$ is identical to the group composition law (10) for $I(n)$. Explicitly, for $(w_1,R_1),(w_2,R_2) \in \mathbb{R}^n \rtimes \mathrm{O}(n)$

$$(w_1,R_1)(w_2,R_2) = (w_1 + \Phi_{R_1}(w_2), R_1R_2) = (w_1 + R_1w_2, R_1R_2). \tag{18}$$

Moreover, since Theorem 1 implies that every isometry can be written as a composition of a translation and an orthogonal transformation, we have the following characterization of the isometry group.

Theorem 2. *The isometry group $I(n)$ is the semidirect product of $\mathbb{R}^n$ and $\mathrm{O}(n)$, i.e., $I(n) \cong \mathbb{R}^n \rtimes \mathrm{O}(n)$.*

Corollary 1. *$\mathbb{R}^n$ is a normal subgroup of of the isometriy group $I(n)$.*

Let $p : N \rtimes K \to K$ be the canonical surjection map, i.e., $(n,k) \mapsto k$. Then p is a group homomorphism. We verify this by taking any $(n_1,k_1),(n_2,k_2) \in N \rtimes K$ and noting that

$$\begin{aligned} p((n_1,k_1)(n_2,k_2)) &= p(n_1\Phi_{k_1}(n_2),k_1k_2) = k_1k_2 \\ &= p((n_1,k_1))p((n_2,k_2)). \end{aligned} \tag{19}$$

Moreover, the kernel of p is N. This can be verified by noting that any (n, e_K) is mapped to e_K under p, therefore $N \subset \ker p$. Conversely, $(n,k) \in \ker p$ implies $k = e_K$ and $(n,k) = (n,e_K) \in N$. Now, the first isomorphism theorem gives us $K \cong (N \rtimes K)/N$.

Proposition 7. *Let $G = N \rtimes K$ be a semidirect product of groups N and G. Then $G/N \cong K$.*

Applying the above result to the isometry group gives

Corollary 2. *$I(n)/\mathbb{R}^n \cong \mathrm{O}(n)$.*

1.2 Euclidean motion group

Elements of $I(n)$ of the form $T_w \circ R$ where $R \in \mathrm{SO}(n)$ are called orientation preserving isometries of $\mathbb{R}^n$. Orientation preserving isometries form a subgroup of $I(n)$.

Definition 3 (Euclidean motion group). *The Euclidean motion group in n-dimensions, denoted $M(n)$, is the collection of all orientation preserving isometries of $\mathbb{R}^n$.*

Just like in case of $I(n)$, we have: $M(n) \cong \mathbb{R}^n \rtimes \mathrm{SO}(n)$. Moreover, $\mathbb{R}^n$ is a normal subgroup of $M(n)$ and if $p : M(n) \to \mathrm{SO}(n)$ is the canonical surjection map, then p is a homomorphism with $\ker(p) = \mathbb{R}^n$ and $\mathrm{SO}(n) \cong M(n)/\mathbb{R}^n$.

In the rest of this report we shall focus on the Euclidean motion group in two dimensions, i.e., $n = 2$.

In particular, as $\mathbb{R}^2 \cong \mathbb{C}$ and $\mathrm{SO}(2) \cong \mathrm{U}(1) \cong \mathbb{T}$, where $\mathbb{T}$ is the 1-dimensional torus, we can make the identification $M(2) = \mathbb{C} \rtimes \mathbb{T}$. With this identification $M(2)$ is the group of orientation preserving isometries of $\mathbb{C}$. We shall denote elements of $M(2)$ by $g(z,\alpha) = t(z) \circ r(\alpha)$, where, for $w \in \mathbb{C}$, $t(z)(w) = w + z$ is a translation and $r(\alpha)(w) = we^{i\alpha}$ is the action of $\mathrm{U}(1)$ on $\mathbb{C}$.

The Euclidean inner product on $\mathbb{C}$ is given by $\langle z, w\rangle = \mathrm{Re}(z\overline{w})$.

$M(2)$ can be embedded in $\mathrm{GL}(2,\mathbb{C})$ as the subgroup with matrices of the form

$$M(2) = \left\{ \begin{bmatrix} e^{i\alpha} & z \\ 0 & 1 \end{bmatrix} \middle| \text{ for any } \alpha \in \mathbb{R} \text{ and } z \in \mathbb{C} \right\}. \tag{20}$$

Hence, $M(2)$ is a linear Lie group. In fact, $I(2)$ is a 3-dimensional Lie group with two connected components. $M(2)$ is the connected component of $I(2)$ containing the identity.

For future reference, we note the following relations.

1. $r(\alpha)r(\beta) = r(\alpha+\beta)$; $r(\alpha)^{-1} = r(-\alpha)$
2. $t(z)t(w) = t(z+w)$; $t(z)^{-1} = t(-z)$
3. $g(z,\alpha) = t(z)r(\alpha)$
4. $g(z,\alpha)^{-1} = g(-r(-\alpha)z, -\alpha)$
5. $g(z,\alpha)g(w,\beta) = g(z + r(\alpha)w, r(\alpha+\beta))$

2 Irreducible representations of $M(2)$

The compact group $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$ has the normalized Haar measure $dr = d\alpha/2\pi$. We start by looking at a family of unitary representations of $M(2)$.

Theorem 3. *Let $a \in \mathbb{R}^2$. There exists a unitary representation π_a of $M(2)$ on $L^2(\mathbb{T})$ defined by*

$$(\pi_a(g)F)(x) = e^{i\langle z, xa\rangle} F(r(\alpha)^{-1}x), \tag{21}$$

where $g = t(z)r(\alpha)$ and $F \in L^2(\mathbb{T})$.

Proof. We verify that π_a is unitary. For $g = t(z)r(\alpha) \in M(2)$ and $F, F' \in L^2(\mathbb{T})$

$$\langle \pi_a(g)F, \pi_a(g)F' \rangle = \int_{\mathbb{T}} F(r^{-1}x)\overline{F'(r^{-1}s)}dx, \tag{22}$$

and since the measure on $\mathbb{T}$ is left invariant

$$\langle \pi_a(g)F, \pi_a(g)F' \rangle = \int_{\mathbb{T}} F(x)\overline{F'(x)}dx = \langle F, F' \rangle. \tag{23}$$

We also verify that π_a as defined above is a group homomorphism of $M(2)$ into $GL(L^2(\mathbb{T})$. Let $g_1 = t(z_1)r(\alpha_1)$ and $g_1 = t(z_2)r(\alpha_2)$, then for $F \in L^2(\mathbb{T})$

$$\begin{aligned}(\pi_a(g_1)\pi_a(g_2)F)(x) &= \pi_a(g_1)e^{i\langle z_2, r(\alpha_2)^{-1}xa\rangle}F(r(\alpha_2)^{-1}x)\\ &= e^{i\langle z_1 + r(\alpha_2)z_2, xa\rangle}F(r(\alpha_1+\alpha_2)^{-1}x)\\ &= (\pi_a(g_1g_2)F)(x).\end{aligned}$$

Finally, we need to show that the mapping $g \mapsto \pi_a(g)$ from $M(2)$ to $GL(L^2(\mathbb{T}))$, with strong operator topology on $GL(L^2(\mathbb{T}))$, is continuous. It is sufficient to prove that the map is continuous at identity, i.e., given $\epsilon > 0$ there exists a neighbourhood U of e in $M(2)$ such that

$$||\pi_a(g)F - F|| < \epsilon \quad \text{for any } g \in U. \tag{24}$$

Since the case of $F = 0$ is trivial, assume $F \neq 0$. We can assume $\epsilon/3 < ||F||$, and since $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$, there exists $\phi \in C(\mathbb{T})$ satisfying

$$||F - \phi|| < \epsilon/3. \tag{25}$$

As $||F|| > \epsilon/3$, $\phi \neq 0$. Since ϕ is a continuous function on the compact group $\mathbb{T}$, it is bounded and uniformly continuous and therefore translations of ϕ are continuous, i.e., there exists a neighbourhood V of identity in $\mathbb{T}$ such that the left regular representation satisfies

$$||L_r\phi - \phi||_\infty < \epsilon/6 \quad \text{for any } r \in V. \tag{26}$$

Moreover, since $|\langle w, ta\rangle| \leq |w||a|$ for an $t \in \mathbb{T}$, there exists a neighbourhood W of 0 in $\mathbb{R}^2$ such that

$$|e^{i\langle w, ta\rangle} - 1| < \frac{\epsilon}{6||\phi||_\infty} \tag{27}$$

for every $w \in W$ and $t \in \mathbb{T}$. Let $U = t(W) \times V$. Then U is a neighbourhood of e in

$M(2)$. If $g = t(z)r(\alpha) \in U$, then we have

$$\begin{aligned}
||\pi_a(g)\phi - \phi|| &= \sup_{t\in\mathbb{T}} \left| e^{i\langle z,ta\rangle}\phi(r^{-1}t) - \phi(t) \right| \\
&\le \left\| e^{i\langle z,ta\rangle}(\phi(r^{-1}t) - \phi(t)) \right\|_\infty + \left\| (e^{i\langle z,ta\rangle} - 1)\phi(t) \right\|_\infty \\
&\le ||L_r\phi - \phi||_\infty + \left\| e^{i\langle z,ta\rangle} - 1 \right\|_\infty ||\phi||_\infty \\
&< \epsilon/6 + \epsilon/6 = \epsilon/3. \qquad (28)
\end{aligned}$$

Finally, using the above inequality, (25) and the relations $||\pi_a(g)f|| = ||f||$, $||\phi|| \le ||\phi||_\infty$ we have

$$\begin{aligned}
||\pi_a(g)F - F|| &\le ||\pi_a(g)F - \pi_a(g)\phi|| + ||\pi_a(g)\phi - \phi|| + ||\phi - F|| \\
&< ||\pi_a(g)(F - \phi)|| + \epsilon/3 + \epsilon/3 \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 < \epsilon. \qquad (29)
\end{aligned}$$

□

The next result shows that the right regular representation of $\mathbb{T}$ intertwines π_a with π_{ra}.

Theorem 4. *Let R_r ($r \in \mathbb{T}$) be the right regular representation of $\mathbb{T}$. Then we have*

$$R_r \circ \pi_a(g) = \pi_{ra}(g) \circ R_r \qquad (30)$$

Proof. Let $r_0 \in \mathbb{T}$, $g = t(z)r \in M(2)$ and $F \in L^2(\mathbb{T})$, we have

$$\begin{aligned}
(R_{r_0}\pi_a(g)F)(t) &= \pi_a(g)F(tr_0) = e^{i\langle z,atr_0\rangle}F(r^{-1}tr_0) \\
&= \pi_a(g)F(tr_0) = (\pi_a(g)R_{r_0})F(t) \qquad (31)
\end{aligned}$$

□

Since R_r is unitary and always non-trivial, the above result shows that if $|a| = |b|$, then π_a and π_b are unitarily equivalent, i.e., $\pi_a \cong \pi_b$, and it is sufficient to study representations π_a for which $a \ge 0$.

An explicit computation of the representation in Theorem 3 gives

Proposition 8. *For $g = t(\rho e^{i\phi})r(\alpha)$ the unitary operator $\pi_a(g)$ ($a \ge 0$) is represented by*

$$(\pi_a(g)F)(\theta) = e^{ia\rho\cos(\phi-\theta)}F(\theta - \alpha). \qquad (32)$$

Proof. As $\langle z,a\rangle = \operatorname{Re} z\bar{a}$, we have $\langle z, r(\theta)a\rangle$. With the given g, we have $z = \rho e^{i\theta}$ and

$$\langle z, r(\theta)a\rangle = \operatorname{Re}(a\rho e^{i(\phi-\theta)}) = a\rho\cos(\phi - \theta). \qquad (33)$$

By Theorem 3 we get

$$\begin{aligned}(\pi_a(g)F)(\theta) &= e^{i\langle z, r(\theta)a\rangle}F(\theta-\alpha)\\ &= e^{ia\rho\cos(\phi-\theta)}F(\theta-\alpha). \end{aligned} \tag{34}$$

□

Proposition 9. *Let $F\in L^2(\mathbb{T})$. Then $\pi_a(r)F=F$ for every $r\in\mathbb{T}$ if and only if F is a constant function.*

Proof. We notice that $\pi_a(r)=L_r$, the left regular representation of $\mathbb{T}$ for every $r\in\mathbb{T}$. Write $F(\theta)=\sum_{n\in\mathbb{Z}}c_n\chi_n(\theta)$ and

$$\pi_a(r(\alpha))\left(\sum_{n\in\mathbb{Z}}c_n\chi_n\right)(\theta)=\sum_{n\in\mathbb{Z}}c_n e^{-in\alpha}\chi_n(\theta). \tag{35}$$

Hence, $F\in L^2(\mathbb{T})$ satisfies $\pi_a(g)F=F$ for every $\theta\in\mathbb{T}$ if and only if $c_n=e^{-in\alpha}c_n$ for every $n\in\mathbb{Z}$. This means $c_n=0$ for $n\neq 0$ and $F=c_0\chi_0=c_0$, a constant. □

We now give a converse to Theorem 4.

Lemma 1. *Let $\phi_a(g)=\langle\pi_a(g)1,1\rangle$, where 1 denotes the constant function that is 1 everywhere on $\mathbb{T}$. Then we have $\phi_a(g)=J_0(a\rho)$ for $g=t(\rho e^{i\phi})r(\alpha)$, where J_0 is the Bessel function of order 0.*

Proof. By Proposition 8 we have

$$\begin{aligned}\phi_a(t(\rho e^{i\phi})r(\alpha)) &= \langle\pi_a(t(\rho e^{i\phi})r(\alpha)1,1\rangle\\ &= \frac{1}{2\pi}\int_0^{2\pi}e^{ia\rho\cos(\phi-\theta)}d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}e^{ia\rho\cos\theta}d\theta\\ &= J_0(a\rho), \end{aligned} \tag{36}$$

where the last equality follows from the definition of the Bessel function. □

Theorem 5. *Let $a,b\in\mathbb{R}^2$. Then π_a is equivalent to π_b if and only if $|a|=|b|$.*

Proof. It is sufficient to prove that if $\pi_a\cong\pi_b$ for $a,b\geq 0$, then $a=b$.

If $\pi_a\cong\pi_b$, then there exists a unitary operator T on $L^2(\mathbb{T})$ such that $T\pi_a(g)=\pi_b(g)T$ for all $g\in M(2)$. If we denote by 1 the constant function with value 1 on $\mathbb{T}$, then $(\pi_b(r)T)(1)=(T\pi_a(r))(1)=T(1)$ for every $r\in\mathbb{T}$. Hence, $T(1)=c$, a constant, by Proposition 9. Since T is unitary, $|c|=1$. We have

$$\begin{aligned}\phi_a(g) &= \langle \pi_a(g)1, 1\rangle = \langle T\pi_(g)1, T1,\rangle \\ &= \langle \pi_b(g)T1, T1\rangle = |c|^2\langle \pi_b(g)1, 1\rangle \\ &= \phi_b(g) \quad \text{for any } g\in M(2). \end{aligned} \tag{37}$$

By Lemma 1, we get

$$J_0(a\rho)=J_0(b\rho) \quad \text{for every } \rho \in \mathbb{R}\,. \tag{38}$$

We know that

$$J_0(x)=\frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}d\theta, \tag{39}$$

differentiating twice gives

$$J_0''(0)=\frac{-1}{2\pi}\int_0^{2\pi}\cos^2\theta d\theta<0. \tag{40}$$

If we put $f_a(x)=J_0(ax)$, then $f_a''(0)=a^2J_0''(0)$. As $J_0(a\rho)=J_0(b\rho)$ we have $a^2J_0''(0)=b^2J_0''(0)$ and hence $a^2=b^2$.

We have shown that if $\pi_a\cong\pi_b$ then $|a|=|b|$. □

This next result shows that the representations π_a are irreducible for $|a|>0$.

Lemma 2. *The limit*

$$\lim_{x\to 0}\left\|\frac{e^{iaxf(\theta)}-1}{x}-iaf(\theta)\right\|=0 \tag{41}$$

holds.

Proof.

$$\left\|\frac{e^{iaxf(\theta)}-1}{x}-iaf(\theta)\right\|_\infty=\frac{1}{|x|}\left\|e^{iaxf(\theta)}-(1+iaxf(\theta))\right\|_\infty \tag{42}$$

By Taylor expansion of e^x

$$e^{iaxf(\theta)}=1+iaxf(\theta)+\frac{(iaxf(\theta))^2}{2!}+o(x^3), \tag{43}$$

we get

$$\left\|\frac{e^{iaxf(\theta)}-1}{x}-iaf(\theta)\right\|_\infty\le\frac{1}{|x|}\left\|\frac{(iaxf(\theta))^2}{2}\right\|\infty\le\frac{a^2}{2}|x|. \tag{44}$$

Hence, the limit holds. □

Theorem 6. *If $|a|>0$, the unitary representation $(\pi_a, L^2(\mathbb{T}))$ is irreducible.*

Proof. By Theorem 4, we can assume $a > 0$ without loss of generality. To show irreducibility, it is sufficient to prove that if a projection operator P on $L^2(\mathbb{T})$ satisfies

$$P\pi_a(g) = \pi_a(g)P \quad \text{for every } g \in M(2), \tag{45}$$

then $P = 0$ or $P = \mathbb{1}$, i.e., $L^2(\mathbb{T})$ has no non-trivial invariant subspaces.

For $n \in \mathbb{Z}$, put $\chi_n(\theta) = e^{in\theta}$ and define $f_n := P\chi_n$. For $g = t(0)r(\alpha) \in M(2)$ we have $(\pi_a(r(\alpha))f_n)(\theta) = f_n(\theta - \alpha)$ and therefore

$$\begin{aligned} f_n(\theta-\alpha) &= (\pi_a(r(\alpha))P\chi_n)(\theta) = (P\pi_a(r(\alpha)))(\theta) \\ &= P\chi_n(\theta-\alpha) = P(e^{in(\theta-\alpha)}) = e^{-in\alpha}f_n(\theta). \end{aligned} \tag{46}$$

With $\theta = \alpha$, we conclude$f_n(\alpha) = c_n e^{in\alpha}$, where $c_n = f_n(0)$ is a constant, for every $n \in \mathbb{Z}$. For $x \in \mathbb{R}$ we have (by Proposition 8)

$$\begin{aligned} P(e^{iax\cos\theta}e^{in\theta}) &= (P\pi_a(t(x)))(e^{in\theta}) = (\pi_a(t(x))P\chi_n)(\theta) \\ &= c_n(\pi_a(t(x))\chi_n)(\theta) = c_n e^{iax\cos\theta}e^{in\theta} \end{aligned} \tag{47}$$

and therefore

$$P\left(\frac{e^{iax\cos\theta}-1}{x}e^{in\theta}\right) = c_n\frac{e^{iax\cos\theta}-1}{x}e^{in\theta}. \tag{48}$$

By Lemma 2 we get $P(e^{in\theta}\cos\theta) = c_n e^{in\theta}\cos\theta$. Similarly, we have $P(e^{iax\sin\theta}e^{in\theta}) = c_n e^{iax\cos\theta}e^{in\theta}$ and by Lemma 2 we get $P(e^{in\theta}\sin\theta) = c_n e^{in\theta}\sin\theta$. Together these prove

$$\begin{aligned} P\chi_{n+1} &= P(e^{i\theta}e^{in\theta}) = P(e^{in\theta}\cos\theta + ie^{in\theta}\sin\theta) \\ &= c_n e^{in\theta}\cos\theta + ic_n e^{in\theta}\sin\theta = c_n\chi_{n+1} \quad \text{for every } n \in \mathbb{Z}\,. \end{aligned} \tag{49}$$

Hence, $c_{n+1} = c_n$ and $c_n = c_0$ for every $n \in \mathbb{Z}$. Since χ_n is an orthonormal basis for $\mathbb{T}$, we get $P = c_0\mathbb{1}$, and as P is a projection operator $P^2 = P \implies c_0^2 = c_0$, therefore $c_0 = 0$ or $c_0 = 1$. We have proved that $P = 0$ or $P = \mathbb{1}$. □

The family of representations $P = \{\pi_a | a > 0\}$ is called the *principal series* of irreducible representations of $M(2)$.

All representations in the principal series are infinite dimensional. We will now look at some one-dimensional representations of $M(2)$. Let $p : M(2) \to \mathbb{T}$ be the canonical projection of $M(2) = \mathbb{C} \rtimes \mathbb{T}$ onto $\mathbb{T}$. Any irreducible representation χ of $\mathbb{T}$ defines a unitary representation $\chi \circ p$ of $M(2)$. The unitary dual of $\mathbb{T}$ is

$$\widehat{\mathbb{T}} = \{\chi_n : r(\alpha) \mapsto e^{in\alpha} | n \in \mathbb{Z}\}. \tag{50}$$

All irreducible representations of $\mathbb{T}$ are one-dimensional, therefore the representations $\chi_n \circ p$ of $M(2)$ are also irreducible.

Remarkably, these one-dimensional representations are the only irreducible unitary representations of $M(2)$ other than the principal series.

Theorem 7. *Any irreducible unitary representation π of the Euclidean motion group $M(2)$ is equivalent to one of the elements in the set*

$$\widehat{M(2)} = \{\pi_a | a > 0\} \cup \{\chi_n \circ p | n \in \mathbb{Z}\}. \tag{51}$$

No two elements of $\widehat{M(2)}$ are equivalent to each other.

3 Fourier transforms

Proposition 10. *Let $g = t(z)r(\alpha) = t(x+iy)r(\alpha)$. Then $dg = dzdr = dxdydr$ is a left invariant Haar measure on $M(2)$. It is also right invariant and $M(2)$ is a unimodular group.*

Proof. We have $t(z_1)r_1t(z_2)r_2 = t(z_1 + r_1z_2)r_1r_2$ and the Lebesgue measure dz on $\mathbb{C}$ is invariant under rigid motion $z_1 \mapsto z_1 + r_1z_2$. We have already seen that dr is the Haar measure on $\mathbb{T}$, therefore we have $d(g_1g_2) = dg_2$. Similarly, $d(g_2, g_1) = dg_2$ and $M(2)$ is a unimodular group. □

The measure on $M(2)$ given by $dg = dzd\alpha/(2\pi)^2 = dm(z)dr$ is called the *normalized Haar measure* on G.

Definition 4 (Fourier transform). *The* Fourier transform *$\widehat{f}$ of a function $f \in L^1(M(2))$ is a function on $R_+^* = (0, \infty)$ with values in $B(L^2(\mathbb{T}))$, the Banach space of bounded linear operators on $L^2(\mathbb{T})$, defined by*

$$\widehat{f}(a) = \int_{M(2)} f(g)\pi_a(g^{-1})dg \quad \text{for } a > 0, \tag{52}$$

where π_a is a principal series unitary representation.

Proposition 11. *If f and h are integrable function on $M(2)$, then we have*

1. $\left\|\widehat{f}(a)\right\| \le ||f||_1$*, for any* $a > 0$
2. $\widehat{f * h} = \widehat{h}\widehat{f}$*, and*
3. $\widehat{(f^*)}(a) = (\widehat{f}(a))^*$

Proof. 1. Let $u \in L^2(\mathbb{T})$. We have

$$\left\|\widehat{f}(a)u\right\| = \left\|\int_{M(2)} f(g)\pi_a(g^{-1})u\,dg\right\| \le \int_{M(2)} |f(g)| \left\|\pi_a(g^{-1})u\right\| dg, \tag{53}$$

and since $\pi_a(g^{-1})$ is unitary, and dg is the normalized Haar measure

$$\left\|\widehat{f}(a)u\right\| \le ||f||_1||u||. \tag{54}$$

2. We use Fubini's theorem and right invariance of dg to simplify $\widehat{f*h}(a)$

$$
\begin{aligned}
\widehat{f*h}(a) &= \int_{M(2)} f*h(g)\pi_a(g^{-1})dg \\
&= \int_{M(2)} \left[\int_{M(2)} f(gs^{-1})h(s)ds \right] \pi_a(g^{-1})dg \\
&= \int_{M(2)} \left[\int_{M(2)} f(gs^{-1})\pi_a(g^{-1})dg \right] h(s)ds \\
&= \int_{M(2)} \left[\int_{M(2)} f(g)\pi_a(s^{-1}g^{-1})dg \right] h(s)ds \\
&= \int_{M(2)} h(s)\pi_a(s^{-1})ds \int_{M(2)} f(g)\pi_a(g^{-1})dg \\
&= \widehat{h}(a)\widehat{f}(a) = \widehat{h}\widehat{f}(a).
\end{aligned} \tag{55}
$$

3. Let $u, v \in L^2(\mathbb{H})$. We have

$$
\begin{aligned}
\langle \widehat{f^*}(a)u, v\rangle &= \int_{M(2)} \langle f^*(g)\pi_a(g^{-1})u, v\rangle dg \\
&= \int_{M(2)} \langle \overline{f(g^{-1})}\pi_a(g^{-1})u, v\rangle dg \\
&= \int_{M(2)} \langle u, f(g^{-1})\pi_a(g)v\rangle dg \\
&= \langle u, \widehat{f}(a)v\rangle = \langle (\widehat{f}(a))^* u, v\rangle
\end{aligned} \tag{56}
$$

Hence $\widehat{f^*}(a) = \widehat{f}(a)^*$.

□

We define the notion of a rapidly decreasing function analogously to the case of $\mathbb{R}^n$.

Definition 5. *A complex valued C^∞-function f on $M(2)$ is called* rapidly decreasing *if for any $N \in \mathbb{N}$ and $m \in \mathbb{N}^3$ we have*

$$
p_{N,m}(f) = \sup_{\alpha \in \mathbb{R}, z \in \mathbb{C}} \left| (1+|z|^2)^N (D^m f)(z, \alpha) \right| < \infty, \tag{57}
$$

where

$$
D^m = \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^{m_1} \left(\frac{1}{i}\frac{\partial}{\partial y}\right)^{m_2} \left(\frac{1}{i}\frac{\partial}{\partial \alpha}\right)^{m_3}. \tag{58}
$$

The vector space of all rapidly decreasing functions on a group G is denoted by $\mathscr{S}(G)$.

The following result shows that $\widehat{f}(a)$ is an integral operator on $L^2(\mathbb{T})$ with its kernel given by

$$
k_f^a(s, r) = \int_{\mathbb{R}^2} f(z, rs^{-1}) e^{-i\langle z, ra\rangle} dm(z). \tag{59}
$$

Proposition 12. *If f is a rapidly decreasing function on $M(2)$ and k_f^a is as defined above, then we have*

$$(\widehat{f}(a)F)(s) = \int_{\mathbb{T}} k_f^a(s,r)F(r)dr \tag{60}$$

for any $a > 0$ and $F \in L^2(\mathbb{T})$.

Proof. Let $F, F' \in L^2(\mathbb{T})$ and $g \in M(2)$ with $g = t(z)r$ and $g^{-1} = t(-r^{-1}z)r^{-1}$. We have

$$\begin{aligned}
\langle \widehat{f}(a)F, F'\rangle &= \int_{M(2)} f(g)\langle \pi_a(g^{-1})F, F'\rangle dg \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{T}} f(z,r)\langle \pi_a(t(-r^{-1}z)r^{-1})F, F'\rangle dm(z)dr \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{T}} f(z,r)\left[\int_{\mathbb{T}} e^{-i\langle r^{-1}z, sa\rangle} F(rs)\overline{F'(s)}ds\right] dm(z)dr \\
&= \int_{\mathbb{T}}\int_{\mathbb{T}}\left[\int_{\mathbb{R}^2} f(z, rs^{-1})e^{-i\langle z, ra\rangle} dm(z)\right] F(r)\overline{F'(s)}dsdr \\
&= \int_{\mathbb{T}}\left[\int_{\mathbb{T}} k_f^a(s,r)F(r)dr\right]\overline{F'(s)}ds.
\end{aligned}$$

□

If we denote the ordinary Fourier transform of $z \mapsto f(z,r)$ by $\widetilde{f}(\xi, r)$:

$$\widetilde{f}(\xi, r) = \int_{\mathbb{R}^2} f(z,r)e^{-i\langle z,\xi\rangle} dm(z), \tag{61}$$

the kernel k_f^a is given by

$$k_f^a(s,r) = \widetilde{f}(ra, rs^{-1}). \tag{62}$$

3.1 Fourier inversion formula

Definition 6. *A bounded linear operator A on a separable Hilbert space H is said to be of* trace class *if for any orthonormal basis $(\phi_n)_{n\in\mathbb{N}}$ of H, the series*

$$\sum_{n\in\mathbb{N}} \langle A\phi_n, \phi_n\rangle \tag{63}$$

converges to a finite sum which is independent of the choice of (ϕ_n).

The sum in (63) is called the *trace* of A and is denoted by $\mathrm{Tr}A$.

If A is of trace class then the series (63) converges absolutely, because the sum is invariant under a change of ordering of ϕ_ns.

The following result gives a sufficient condition for an operator to be of trace class.

Proposition 13. *Let H be a separable Hilbert space. If a bounded linear operator A on*

H satisfies

$$\sum_{n,m\in\mathbb{N}} |\langle A\phi_n, \phi_m\rangle| < \infty \tag{64}$$

for a fixed orthonormal basis $(\phi_n)_{n\in\mathbb{N}}$*, then* A *is of trace class. Moreover, if* U *and* V *are two bounded operators on* H*, then* UAV*,* AVU*, and* VUA *are of trace class and have the same trace.*

Proof. Let $a_{m,n} = \langle A\phi_m, \phi_n\rangle$, $u_{m,n} = \langle U\phi_m, \phi_n\rangle$, and $v_{n,m} = \langle V\phi_m, \phi_n\rangle$. We have

$$A\phi_m = \sum_{n=0}^{\infty} a_{m,n}\phi_n, \quad U\phi_m = \sum_{n=0}^{\infty} u_{m,n}\phi_n \quad \text{and} \quad V\phi_m = \sum_{n=0}^{\infty} v_{m,n}\phi_n.$$

Hence

$$UAV\phi_m = \sum_{n,k,l=0}^{\infty} u_{m,n}a_{n,k}v_{k,l}\phi_l, \tag{65}$$

and with Schwarz inequality in l^2 gives

$$\begin{aligned}\sum_{m=0}^{\infty} |u_{m,n}a_{n,k}v_{k,m}| &\le |a_{n,k}| \left(\sum_{m=0}^{\infty} |u_{m,n}|^2\right)^{1/2} \left(\sum_{m=0}^{\infty} |u_{k,m}|^2\right)^{1/2} \\ &= |a_{n,k}| ||U^*\phi_n|| ||V^*\phi_k|| j \le |a_{n,k}| ||U^*|| ||V^*||\end{aligned} \tag{66}$$

and finally we have

$$\sum_{m=0}^{\infty} |\langle UAV\phi_m, \phi_m\rangle| \le \sum_{n,k=0}^{\infty} |a_{n,k}| ||U^*|| ||V^*|| < \infty, \tag{67}$$

by hypothesis. Hence, the series $\sum_{m,n,k} u_{m,n}a_{n,k}v_{k,m}$ converges absolutely and we can change the order of terms freely. We use Parseval's equality to get

$$\begin{aligned}\sum_m \langle UAV\phi_m, \phi_n\rangle &= \sum_m \langle V\phi_m, A^*U^*\phi_n\rangle \\ &= \sum_{m,n} \langle V\phi_m, \phi_n\rangle\langle\phi_n, A^*U^*\phi_m\rangle \\ &= \sum_{m,n,k} \langle V\phi_m, \phi_n\rangle\langle A\phi_n, \phi_k\rangle\langle U\phi_k, \phi_m\rangle \\ &= \sum_k \langle AVU\phi_k, \phi_k\rangle = \sum_n \langle VUA\phi_n, \phi_n\rangle.\end{aligned} \tag{68}$$

Let $(\psi_n)_{n\in\mathbb{N}}$ be another orthonormal basis of H, then the operator W, defined by $W\phi_n = \psi_n$ is unitary and

$$\sum_{m=0}^{\infty} \langle UAV\psi_m, \psi_m\rangle = \sum_{m=0}^{\infty} \langle UAV\phi_m, \phi_m\rangle, \tag{69}$$

Hence, UAV is of trace class, and in particular, putting $U = V = \mathbb{1}$ shows that A is of trace class. Similarly, AVU and VUA are also of trace class and (68) shows that $\mathrm{Tr}(UAV) = \mathrm{Tr}(AVU) = \mathrm{Tr}(VUA)$. □

Proposition 14. *If $k(\theta,\phi)$ is a C^2-function on $\mathbb{T}^2$, then the Fourier series*

$$\sum_{m,n\in\mathbb{Z}} a_{m,n} e^{i(m\theta+n\phi)} \tag{70}$$

of k, where

$$a_{m,n} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} k(\theta,\phi)e^{-i(m\theta+n\phi)}d\theta d\phi, \tag{71}$$

converges absolutely and uniformly.

Proposition 15. *If $k(\theta,\phi)$ is a C^2-function on $\mathbb{T}^2$, then the operator L on $L^2(\mathbb{T})$ defined by*

$$(LF)(\theta) = \frac{1}{2\pi}\int_0^{2\pi} k(\theta,\phi)F(\phi)d\phi \tag{72}$$

is of trace class with trace

$$\operatorname{Tr} L = \frac{1}{2\pi}\int_0^{2\pi} k(\theta,\theta)d\theta. \tag{73}$$

Proof. Let $\chi_n(\theta) = e^{in\theta}$, then $(\chi_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T})$. By the previous proposition, the Fourier series of k converges and uniformly. We have

$$\langle L\chi_m, \chi_n\rangle = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} k(\theta,\phi)e^{i(m\phi-n\theta)}d\phi d\theta = a_{m,-n}, \tag{74}$$

and it follows that

$$\sum_{m,n\in\mathbb{Z}} |\langle L\chi_m,\chi_n\rangle| < \infty \tag{75}$$

and by Proposition 13, L is of trace class. As the series $k(\theta,\theta)$ converges uniformly on $\mathbb{T}$, we can integrate it term by term and get

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} k(\theta,\theta)d\theta &= \sum_{m,n\in\mathbb{Z}} a_{m,n}\int_0^{2\pi} e^{i(m+n)\theta}d\theta \\ &= \sum_{m,n\in\mathbb{Z}} a_{m,n}\delta_{m+n,0} \\ &= \sum_{m\in\mathbb{Z}} a_{m,-m} = \sum_{m\in\mathbb{Z}}\langle L\chi_m,\chi_m\rangle = \operatorname{Tr} L.\end{aligned} \tag{76}$$

□

The above series of propositions culminates in the Fourier inversion formula for rapidly decreasing functions.

Theorem 8 (inversion formula)**.** *Any function $f \in \mathscr{S}(M(2))$ may be recovered from its Fourier transform $\widehat{f}$ by the formula*

$$f(g) = \int_0^\infty \operatorname{Tr}\Big(\pi_a(g)\widehat{f}(a)\Big)a\,da. \tag{77}$$

In particular, $\pi_a(g)\widehat{f}(a)$ and $\widehat{f}(a)$ are of trace class.

Proof. By Proposition 12 $\widehat{f}(a)$ can be seen as an integral operator with a smooth kernel, and therefore we can use Proposition 15 to say that $\widehat{f}(a)$ is of trace class. Let $g = t(z)u$ for $u \in \mathbb{T}$; then $\pi_a(g)\widehat{f}(a)$ is an integral operator

$$\begin{aligned}(\pi_a(g)\widehat{f}(a)F)(x) &= e^{i\langle z,xa\rangle}(\widehat{f}(a)F)(u^{-1}s)\\ &= \int_{\mathbb{T}} e^{i\langle z,xa\rangle} k_f^a(u^{-1}x,r)F(r)dr\\ &= \int_{\mathbb{T}} e^{i\langle z,xa\rangle} \widetilde{f}(ra, rx^{-1}u)F(r)dr, \end{aligned} \tag{78}$$

with kernel $m_f^a(g;x,r) = e^{i\langle z,xa\rangle}\widetilde{f}(ra, rs^{-1}u)$. By Proposition 15,

$$\begin{aligned}\mathrm{Tr}\Big(\pi_a(g)\widehat{f}(a)\Big) &= \int_{\mathbb{T}} m_f^a(g;r,r)dr\\ &= \int_{\mathbb{T}} e^{i\langle z,ra\rangle}\widetilde{f}(ra,u)dr \end{aligned} \tag{79}$$

For fixed r, the function $f_r : z \mapsto f(z,r)$ is a rapidly decreasing on $\mathbb{R}^2$, and

$$\widetilde{f}(a,r) = \int_{\mathbb{R}}^2 f(z,r)e^{-i\langle z,a\rangle}dm(z) \tag{80}$$

is the Fourier transform on $\mathbb{R}^2$; hence, we can use the inversion formula for $f_r \in \mathscr{S}(\mathbb{R}^2)$ to get

$$\begin{aligned}f(g) = f(z,u) &= \int_{\mathbb{R}^2} \widetilde{f}(\xi,u)e^{i\langle z,\xi\rangle}dm(\xi)\\ &= \int_0^\infty \int_{\mathbb{T}} \widetilde{f}(ra,u)e^{i\langle z,ra\rangle}adadr\\ &= \int_0^\infty \mathrm{Tr}\Big(\pi_a(g)\widehat{f}(a)\Big)ada, \end{aligned} \tag{81}$$

where in the second line we transformed to polar coordinates, and used Fubini's theorem to get to the last line by integrating over $\mathbb{T}$. □

Proposition 16. *If f and h belong to $\mathscr{S}(M(2))$, then $f * h$ and $f^*(g) = \overline{f(g^{-1})}$ also belong to $\mathscr{S}(M(2))$.*

Next, we want to prove the Parseval equality for Fourier transforms.

3.2 Parseval identity

Let H_1 and H_2 be two separable Hilbert spaces. The set of bounded linear operators from H_1 into H_2 is denoted by $B(H_1, H_2)$. Let $(\phi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of

H_1. For any element A of $B(H_1, H_2)$ put

$$||A||_2^2 = \sum_{n\in\mathbb{N}} ||A\phi_n||^2. \tag{82}$$

Let $(\psi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H_2. If $(\phi'_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H_1, by Parseval's equality we have

$$\begin{aligned}\sum_{n\in\mathbb{N}} ||A\phi_m||^2 &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} |\langle A\phi_n, \psi_m\rangle|^2 = \sum_{m\in\mathbb{N}} ||A^*\psi_m||^2 \\ &= \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} \left|\langle A^*\psi_m, \phi'_n\rangle\right|^2 = \sum_{n\in\mathbb{N}} \left\|A\phi'_n\right\|^2.\end{aligned} \tag{83}$$

Hence, $||A||_2$ is independent of the choice of basis (ϕ_n). Moreover, the above calculation also shows that $||A||_2 = ||A^*||_2$.

Definition 7. *An operator $A \in B(H_1, H_2)$ is called a* Hilbert-Schmidt operator *if $||A||_2 < \infty$. The set of Hilbert-Schmidt operators is denoted by $B_2(H_1, H_2)$.*

$B_2(H_1, H_2)$ is a subspace of $B(H_1, H_2)$ and $||\cdot||_2$ is a norm on $B_2(H_1, H_2)$ called the *Hilbert-Schmidt* norm. Moreover, if $A, B \in B_2(H_1, H_2)$, then the inner product

$$\langle A, B\rangle = \sum_{n\in\mathbb{N}} \langle A\phi_n, B\phi_n\rangle \tag{84}$$

is well defined and $B_2(H_1, H_2)$ becomes a Hilbert space. From the above definition it is easy to see that B^*A is a trace class operator on H_1 with $\operatorname{Tr} B^*A = \langle A, B\rangle$. As a consequence we have the following.

Proposition 17. *$A \in B(H_1, H_2)$ is a Hilbert-Schmidt operator if and only if A^*A is a trace class operator on H_1.*

We can now prove Parseval's equality for Fourier transforms.

Theorem 9 (Parseval's equality)**.** *If f belongs to $\mathscr{S}(M(2))$, then $\widehat{f}(a)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{T})$ and it satisfies*

$$\int_{M(2)} |f(g)|^2 dg = \int_0^\infty \left\|\widehat{f}(a)\right\|_2^2 a da. \tag{85}$$

Proof. Define $h := f * f^*$. By Proposition 16 $h \in \mathscr{S}(M(2))$ is also a rapidly decreasing function. By Proposition 15 $\widehat{h}(a)$ is of trace class and $\operatorname{Tr}\widehat{h}(a) = \operatorname{Tr}\left(\widehat{f^*}(a)\widehat{f}(a)\right) = \operatorname{Tr}\left(\widehat{f}(a)^*\widehat{f}(a)\right)$ (by Proposition 11). Hence, we have $\widehat{h}(a) = \left\|\widehat{f}(a)\right\|_2^2$ and by Proposition 17, $\widehat{f}(a)$ is a Hilbert-Schmidt operator. By Theorem 8 we have

$$h(e) = \int_0^\infty \operatorname{Tr}\left(\widehat{h}(a)\right) a da = \int_0^\infty \left\|\widehat{f}(a)\right\|_2^2 a da, \tag{86}$$

and by definition of the convolution product, we have

$$h(e)=f*f^*(e)=\int_{M(2)} f(g)\overline{f(g)}dg=\int_{M(2)} |f(g)|^2 dg. \tag{87}$$

□

In general, we cannot define the character of the representation π_a as the unitary operator $\pi_a(g)$ need not be of trace class and the series

$$\sum_{n\in\mathbb{Z}} \langle \pi_a(g)\chi_n, \chi_n\rangle \tag{88}$$

may not converge. One way out is to look at the above series as a distribution on $G=M(2)$.

Definition 8 (distribution). *Let $C_c(G)$ be the space of all complex valued C^∞-functions on G with compact support. A continuous linear form on $C_c(G)$ is called a* distribution *on G.*

We define the character χ_a as the linear form

$$\chi_a : f \mapsto \sum_{n\in\mathbb{Z}} \int_G f(g)\langle \pi_a(g)\chi_n, \chi_n\rangle dg = \sum_{n\in\mathbb{Z}} \langle \pi_a^f \chi_n, \chi_n\rangle = \operatorname{Tr} \pi_a^f \tag{89}$$

for $f\in C_c(G)$, where

$$\pi_a^f = \int_G f(g)\pi_a(g)dg. \tag{90}$$

If we let $h(g)=f(g^{-1})$, then $\pi_a^f=\widehat{h}(a)$ and we write $h(re^{i\phi},\alpha)=h[r,\phi,\alpha]$.

We have the following characterization of distributions.

Theorem 10. *For any fixed $a>0$, the linear form $\chi_a : f\mapsto \pi_a^f$ is a distribution on $M(2)$. Moreover, χ_a is equal to $J_0(a|z|)\otimes\delta(\alpha)$, where J_0 is the Bessel function of order 0 and δ is the Dirac measure at 0 on $\mathbb{T}$.*

Proof. By Propositions 12 and 15 π_a^f is of trace class and

$$\begin{aligned}
\operatorname{Tr}\pi_a^f &= \frac{1}{2\pi}\int_0^{2\pi} k_h^a(\theta,\theta)d\theta \\
&= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi} h[r,\phi,0]e^{iar\cos(\phi-\theta)} r\,dr\,r\,d\phi\,d\theta \\
&= \int_{\mathbb{R}^2} f(-z,0)J_0(a|z|)dm(z) \\
&= \int_{\mathbb{T}}\int_{\mathbb{R}^2} f(z,\alpha)J_0(a|z|)dm(z)d\delta(\alpha).
\end{aligned} \tag{91}$$

□

We now want to extend the Fourier transform uniquely from $\mathscr{S}(M(2))$ to an isometry of $L^2(M(2))$.

3.3 Plancherel theorem

Lemma 3. *Let $H_1 = L^2(X,\mu)$, $H_2 = L^2(Y,\nu)$, and let Φ be the mapping of $H = L^2(X \times Y, \mu \times \nu)$ into the space $B_2(H_2, H_1)$ of Hilbert-Schmidt operators which maps $k \in H$ into the integral operator K with the kernel k. Then Φ is an isometry of H onto $B_2(H_2, H_1)$.*

Definition 9. *Let X, Y be two sets and $C(X)$, $C(Y)$ and $C(X \times Y)$ be the vector spaces of all complex valued functions on X, Y and $X \times Y$ respectively. For any two functions $f \in C(X)$ and $g \in C(Y)$, define*

$$(f \circledast g)(x,y) = f(x)g(y) \tag{92}$$

such that $f \circledast g \in C(X \times Y)$.

Since the mapping $(f,g) \mapsto f \circledast g$ is bilinear from $C(X) \times C(Y)$ into $C(X \times Y)$, there exists a linear map $\phi : C(X) \otimes C(Y) \to C(X \times Y)$ such that

$$\phi(f \otimes g) = f \circledast g, \tag{93}$$

ϕ is injective. If we let $\phi(h) = 0$, then we have

$$h = \sum_{m,n} a_{m,n} f_m \otimes g_n, \tag{94}$$

where (f_m) and (g_n) are linearly independent families in $C(X)$ and $C(Y)$ respectively. Since

$$\sum_{m,n} a_{m,n} f_m(x) g_n(x) = 0 \quad \text{for all } x \in X, y \in Y \tag{95}$$

we have $\sum_m a_{m,n} f_m = 0$ for all n by linearly independence of (g_n). And by linear independence of (f_m), $a_{n,m} = 0$ for all n,m and $h = 0$.

Lemma 4. *Let $H_1 = L^2(X,\mu)$, $H_2 = L^2(Y,\nu)$ and $H = L^2(X \times Y, \mu \times \nu)$. Then the mapping ϕ defined above can be extended uniquely to an isometry Φ of the Hilbert space tensor product $H_1 \otimes H_2$ onto H.*

Due to the isometry Φ, we can identify $f \otimes g$ with $f \circledast g$ and we write $(f \otimes g)(x,y) = f(x)f(y)$.

Theorem 11. *Let $G = M(2)$. Then $\mathscr{S}(G)$ is dense in $L^2(G)$.*

Let $\mathbf{B}_2 = B_2(L^2(\mathbb{T}))$ be the Hilbert space of all Hilbert-Schmidt operators on $L^2(\mathbb{T})$, and put $H_a = \mathbf{B}_2$. Define $H = \int_0^\infty \oplus H_a a da$ and let L be an element in H. Then L is a function on $\mathbb{R}_+ = (0,\infty)$ with values in $\mathbf{B}_2$. Since $L(a)$ (value of L at a) is a Hilbert-Schmidt operator on $L^2(\mathbb{T})$, by Lemma 3 it is an integral operator with kernel

$k_a \in L^2(\mathbb{T}\times\mathbb{T})$. We have

$$\begin{aligned}||L||^2 &= \int_0^\infty ||L(a)||_2^2 a da \\ &= \int_0^\infty ||k_a||_2^2 a da \\ &= \int_0^\infty \int_{\mathbb{T}}\int_{\mathbb{T}} |k_a(s,r)|^2 ds dr a da, \end{aligned} \tag{96}$$

and $\Phi : L \mapsto k_a(s,r)$ is an isometry of H onto $L^2(\mathbb{R}_+ \times \mathbb{T}\times\mathbb{T})$ (again, by Lemma 3). We identify H with $L(\mathbb{R}_+ \times \mathbb{T}\times\mathbb{T})$ by the map Φ.

Let $\phi : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}^2$ be defined by

$$(a,r) \mapsto ra, \tag{97}$$

for $a \in \mathbb{R}_+$ and $r \in \mathbb{T}$. Then the transformation to polar coordinates, $g \mapsto g\circ\phi$, is an isometry of $L^2(\mathbb{R}^2)$ onto $L^2(\mathbb{R}_+,\mathbb{T})$. We identify $L^2(\mathbb{R}^2)$ with $L^2(\mathbb{R}_+\times\mathbb{T})$.

Theorem 12 (Plancherel theorem)**.** *Let $\mathbf{B}_2 = B_2(L^2(\mathbb{T}))$ be the Hilbert space of all Hilbert-Schmidt operators on $L^2(\mathbb{T})$. Put $H_a = \mathbf{B}_2$ for all $a > 0$ and $H = \int_0^\infty \oplus H_a a da$. Then the Fourier transform $\mathscr{F} : f \mapsto \widehat{f}$ can be extended uniquely to an isometry F of $L^2(M(2))$ onto H.*

Proof. Parseval's identity (Theorem 9) shows that $\mathscr{F}$ is an isometry of $\mathscr{S}(M(2))$ into H. Since $\mathscr{S}(M(2))$ is dense in $L^2(M(2))$ (Theorem 11), $\mathscr{F}$ can be extended uniquely to an isometry F of $L^2(M(2))$ into H. Now we need to show that F is surjective.

Since F is an isometry, the image $\operatorname{Im}F$ is closed in H. To prove the surjectivity of F, it is suffices to show that $\operatorname{Im}\mathscr{F}$ is dense in H. Moreover, since $\operatorname{Im}\mathscr{F} \subset \operatorname{Im}F$, it is sufficient to show that for any $k \in H = L^2(\mathbb{R}^2\times\mathbb{T})$ and $\epsilon > 0$, there exists an element f in $\mathscr{S}(M(2))$ such that

$$\left\|k - k_f\right\|_2 < \epsilon. \tag{98}$$

By Lemma 4, we can identify $L^2(\mathbb{R}^2\times\mathbb{T}) = L^2(\mathbb{R}^2)\otimes L^2(\mathbb{T})$ and we can assume that the element k in (98) is of the form $k = g\otimes h$ for $g \in L^2(\mathbb{R}^2)$ and $h \in L^2(\mathbb{T})$. Moreover since the space of trigonometric polynomials is dense $L^2(\mathbb{T})$, we cal assume without loss of generality that $h = \chi_n$ for some $n \in \mathbb{Z}$. Let $u(ra) = \chi_n(r)g(ra)$ for $r \in \mathbb{T}$ and $a > 0$, where $\chi_n(r(\alpha)) = e^{in\alpha}$. Then $u \in L^2(\mathbb{R}^2)$. We shall use the fact that $\mathscr{S}(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$ to get an element $v \in \mathscr{S}(\mathbb{R}^2)$ such tha t

$$||u - v||_2 < \epsilon. \tag{99}$$

Let $\mathscr{F}^* v = w$ be the inverse Fourier transform of v on $\mathbb{R}^2$. Then $w \in \mathscr{S}(\mathbb{R}^2)$ is a rapidly decreasing function. Hence $f = w\otimes\chi_{-n}$ is a rapidly decreasing function on $M(2)$, i.e. $f \in \mathscr{S}(M(2))$. This f satisfies (98). Since

$$k_a^f(s,r) = (\mathscr{F}w)(ra)\chi_{-n}(rs^{-1}) = v(ra)\chi_{-n}(r)\chi_n(s), \tag{100}$$

we have

$$
\begin{aligned}
\left\|k-k_f\right\|_2 &= \left\|g\otimes\chi_n-k_f\right\|_2 \\
&= \|(\chi_{-n}u)\otimes\chi_n-(\chi_{-n}v)\otimes\chi_n\|_2 \\
&= \|\chi_{-n}(u-v)\|_2\|\chi_n\|_2 \\
&= \|u-v\|<\epsilon.
\end{aligned} \tag{101}
$$

□

We can use the Plancherel theorem to decompose the regular representation into irreducible representations and prove an analogue of the Peter-Weyl theorem for $M(2)$.

Proposition 18. *Let π be a unitary representation of a topological group G on a separable Hilbert space H. Let $\mathbf{B}_2 = B_2(H)$ be the Hilbert space of all Hilbert-Schmidt operators on H and define a unitary representation of τ of G on $\mathbf{B}_2$ by setting*

$$\tau(g)(A)=\pi(g)A \quad \textit{for } A\in\mathbf{B}_2 \textit{ and } g\in G. \tag{102}$$

Then τ can be decomposed as the direct sum of countable copies of π. More precisely, let (ϕ_n) be an orthonormal basis of H, P_n be the projection on $\mathbb{C}\phi_n$ and

$$\mathbf{B}_2^n=\{A\in\mathbf{B}_2 \mid AP_n=A\}. \tag{103}$$

Then we have

$$\mathbf{B}_2=\bigoplus_{n=0}^{\infty}\mathbf{B}_2^n \quad \textit{and} \quad \tau\big|_{\mathbf{B}_2^n}\cong\pi \tag{104}$$

for all $n\in\mathbb{N}$.

Finally, we have:

Theorem 13. *Let L be the left regular representation of $G=M(2)$. Then L is decomposed as follows*

$$L\cong\int_0^{\infty}\bigoplus\tau_a a\,da, \tag{105}$$

where τ_a is the direct sum of countable copies of π_a:

$$\tau_a\cong\bigoplus_{n\in\mathbb{Z}}\pi_a \quad \textit{for all } a>0. \tag{106}$$

4 Application to the quantum free particle

We end with a short and relatively informal discussion on the application of the representation theory of $M(2)$ to the problem of a free particle in quantum mechanics.

At the beginning we saw a matrix representation of $M(2)$

$$M(2) = \left\{ \begin{bmatrix} e^{i\alpha} & z \\ 0 & 1 \end{bmatrix} \middle| \text{ for any } \alpha \in \mathbb{R} \text{ and } z \in \mathbb{C} \right\}. \tag{107}$$

From here, we can see that the Lie algebra of $M(2)$ is the real vector space spanned by

$$L = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \tag{108}$$

with Lie bracket relations

$$[L, P_1] = P_2, \quad [L, P_2] = -P_1, \quad [P_1, P_2] = 0. \tag{109}$$

The so called *Schrödinger representation* π_S provides a unitary Lie algebra representation on the space $L^2(\mathbb{R}^2)$. This is given by the operators

$$\pi_S(P_1) = -\frac{\partial}{\partial x_1}, \quad \pi_S(P_2) = -\frac{\partial}{\partial x_2}, \tag{110}$$

and

$$\pi_S(L) = -\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right). \tag{111}$$

Hamiltonian operator for a free particle in two dimensions is

$$\widehat{H} = -\frac{1}{2m} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \tag{112}$$

and solutions to the Schrödinger equation can be found as solutions to the eigenvalue equation

$$\widehat{H} \psi(x_1, x_2) = -\frac{1}{2m} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \psi(x_1, x_2) = E \psi(x_1, x_2). \tag{113}$$

The operators $\pi_S(L)$, $\pi_S(P_1)$ and $\pi_S(P_2)$ commute with $\widehat{H}$ and provide a representation of the Lie algebra of $M(2)$ on the eigenspace of $\widehat{H}$ with eigenvalue E. These eigenspaces are infinite dimensional and are characterized by the non-negative eigenvalue E which has the physical interpretation of energy.

An element $g(r(\alpha), w)$ of $M(2)$ acts on the space of solutions $\psi(x) = \psi(x_1, x_2)$ by

$$g(r(\alpha), w) \cdot \psi(x) = \psi(r(-\alpha)(x - w)), \tag{114}$$

which is very similar to the left translation of the function ψ. In fact, this representation is the same as the exponentiated version of the Schrödinger representation π_S. For a translation $t(w) \in M(2)$,

$$t(w) \cdot \psi(x) = e^{w_1 \pi_S(P_1) + w_2 \pi_S(P_2)} \psi(x) = \psi(x - w), \tag{115}$$

and for a rotation $r(\alpha) \in M(2)$,

$$r(\alpha) \cdot \psi(x) = e^{\pi_S(L)\alpha}\psi(x) = \psi(r(-\alpha)x). \tag{116}$$

By passing over to the "momentum space", these representations can be shown to be equivalent to the principal series of representations described in Section 2.

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